2. (a) $\Omega = \{HH, HT, TH, TT\}$

$X(HH) = X(HT) = 1$, $X(TH) = X(TT) = 0$

$\sigma(X) = \{ \underline{\{X \in A\}} \mid A \subseteq \mathbb{R} \}$

① $0 \notin A$, $1 \notin A$. $\{X \in A\} = \emptyset$

② $0 \notin A$, $1 \in A$. $\{X \in A\} = \{HH, HT\}$

③ $0 \in A$, $1 \notin A$. $\{X \in A\} = \{TH, TT\}$

④ $0, 1 \in A$, $\{X \in A\} = \Omega$

$\sigma(X) = \{ \emptyset, \{HH, HT\}, \{TH, TT\}, \Omega \}$

$\sigma(Y) = \cdots\cdots$

(b), (c) $\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B)$

(*)

$$\mathbb{P}(X \in \{1\}, Y \in \{1\}) = \mathbb{P}(X \in \{1\})\, \mathbb{P}(Y \in \{1\})$$

$$\mathbb{P}(X \in \{1\}, Y \in \{0\})$$

$$= \underbrace{\mathbb{P}(X \in \{1\}) - \mathbb{P}(X \in \{1\}, Y \in \{1\})}_{= \mathbb{P}(X \in \{1\},\, Y \in \mathbb{R} \setminus \{1\})}$$

(*)

$$= \mathbb{P}(X \in \{1\}) - \mathbb{P}(X \in \{1\}) \cdot \mathbb{P}(Y \in \{1\})$$

$$= \mathbb{P}(X \in \{1\}) \cdot (\underline{1 - \mathbb{P}(Y \in \{1\})})$$

$$= \mathbb{P}(X \in \{1\}) \cdot \mathbb{P}(Y \in \{0\})$$

1. (f) $\quad S_0 = 1 \qquad f_1 = \frac{1}{2}, \quad f_2 = 1, \quad f_3 = 2$

$$r > 0, \qquad V_1 = (S_1 - 1)_+$$

$$0 = X_0 = S_0 \cdot \alpha + V_0 \cdot \beta + 1 \cdot (-\alpha \cdot S_0 - \beta V_0)$$

$$X_1 = S_1 \cdot \alpha + V_1 \cdot \beta + (1+r) \cdot (-\alpha \cdot S_0 - \beta V_0)$$

$$= \begin{cases} \underbrace{f_1 \cdot S_0}_{1/2} \cdot \alpha + 0 \cdot \beta + (1+r) \cdot (-\alpha \underbrace{S_0}_{1} - \beta V_0) & 1 \\ \underbrace{f_2 \cdot S_0}_{1} \cdot \alpha + 0 \cdot \beta + (1+r)(-\alpha \underbrace{S_0}_{1} - \beta V_0) & 2 \\ \underbrace{f_3 \cdot S_0}_{2} \cdot \alpha + 1 \cdot \beta + (1+r)(-\alpha \underbrace{S_0}_{1} - \beta V_0) & 3 \end{cases}$$

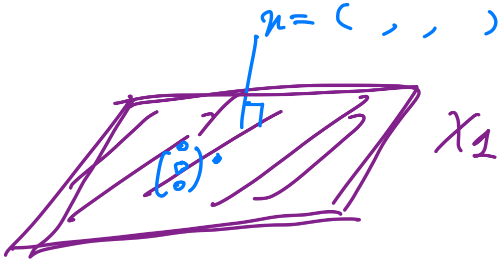
$$X_1 = \alpha \cdot \begin{pmatrix} 1/2 \\ 1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \alpha \begin{pmatrix} 1+r \\ 1+r \\ 1+r \end{pmatrix} - \beta \begin{pmatrix} (1+r)V_0 \\ (1+r)V_0 \\ (1+r)V_0 \end{pmatrix}$$

$$X_1 = \alpha \cdot \begin{pmatrix} \frac{1}{2} - (1+r) \\ 1 - (1+r) \\ 2 - (1+r) \end{pmatrix} + \beta \begin{pmatrix} -(1+r)V_0 \\ -(1+r)V_0 \\ 1 - (1+r)V_0 \end{pmatrix} \in \mathbb{R}^3$$

$$A = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x \geq 0,\ y \geq 0,\ z \geq 0 \right\}.$$

we want to find $V_0 \geq 0$ s.t.

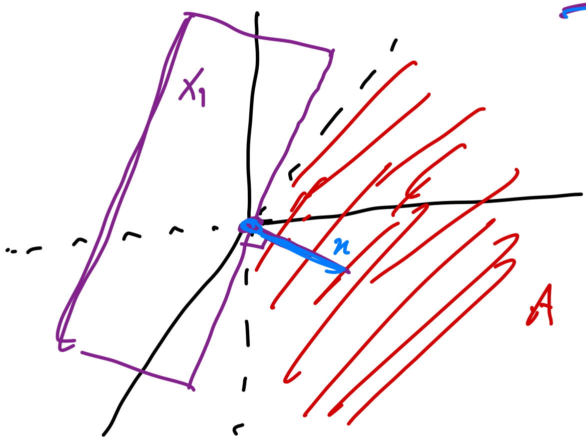
(*) $$X_1 \cap A = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}. \iff X \text{ is arbitrage free.}$$



$$\boxed{\begin{array}{l} 0 \neq \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in A \\ \quad x > 0 \\ \text{or } y > 0 \\ \text{or } z > 0 \end{array}}$$

(*) $\Longleftarrow$ " $n = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ has positive coordinate "

$n_1 > 0$, $n_2 > 0$, $n_3 > 0$



$n = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ is a normal vector of X_1

$$\Longleftrightarrow \quad \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} - (1+r) \\ 1 - (1+r) \\ 2 - (1+r) \end{pmatrix} = 0$$

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \cdot \begin{pmatrix} -(1+r)V_0 \\ -(1+r)V_0 \\ 1-(1+r)V_0 \end{pmatrix} = 0$$

$$\Longleftrightarrow \quad \begin{aligned} n_1 &= \ldots\ldots V_0, r > 0 \\ n_2 &= \ldots\ldots V_0, r > 0 \\ n_3 &= \ldots\ldots V_0, r > 0 \end{aligned}$$

$$\Longleftrightarrow \quad ? < V_0 < ?$$

1.(b)

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ f_1 & f_2 & f_3 \end{pmatrix}}_{A} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 1+r \end{pmatrix}}_{b}$$

1.(d) $S_0 = 1$ $\quad f_1 = \frac{1}{2}$ $\quad f_2 = 1$ $\quad f_3 = 2$

$$X_0 = \alpha \cdot S_0 + \beta \cdot 1$$

$$\downarrow$$

$$X_1 = \alpha \cdot S_1 + \beta(1+r) = \begin{cases} \frac{1}{2}\alpha + (1+r)\beta \\ \alpha + (1+r)\beta \\ 2\alpha + (1+r)\beta \end{cases}$$

$$V_1 = \begin{cases} -1 \\ 0 \\ 2 \end{cases}$$