Last time:

1 No arb

2 $\boxed{\text{AFP}}$

Stock

Bank

$\boxed{\text{NTA (option)}}$

V_0

Market

3 Compute AFP by replication

4.4. Conditional expectation.

Definition 4.28. Let X be a random variable, and $n \leqslant N$. We define $\boldsymbol{E}(X \mid \mathcal{F}_n) = \boldsymbol{E}_n X$ to be the *random variable* given by

$$\boxed{\boldsymbol{E}_n X(\omega) = \sum_{x_i \in \operatorname{Range}(X)} x_i \boxed{\boldsymbol{P}(X = x_i \mid \Pi_n(\omega))}}$$

where

$$\boxed{\Pi_n(\omega)} = \{\omega' \in \Omega \mid \omega'_1 = \omega_1, \ldots, \omega'_n = \omega_n\}$$

Remark 4.29. The above formula does not generalize well to infinite probability spaces. We will develop certain properties of $\boldsymbol{E}_n$, and then only use those properties going forward.

Example 4.30. If we represent Ω as a tree, $\boldsymbol{E}_n X$ can be computed by averaging over leaves.

Remark 4.31. $\boldsymbol{E}_n X$ is the "best approximation" of X given only the first n coin tosses.

$$EX = \sum x_i \boxed{P(X = x_i)}$$

Proposition 4.32. *The conditional expectation* $\boxed{\boldsymbol{E}_n X}$ *defined by the above formula satisfies the following two properties:*

(1) $\boldsymbol{E}_n X$ *is an* $\mathcal{F}_n$*-measurable* random variable.

(2) *For every* $A \in \mathcal{F}_n$, $\displaystyle\sum_{\omega \in A} \boldsymbol{E}_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$.

$P(A)$

$\frac{1}{P(A)}$

Remark 4.33. This property is used to define conditional expectations in the continuous time setting. It turns out that there is exactly one random variable that satisfies both the above properties; and thus we define $\boldsymbol{E}_n X$ to be the unique random variable which satisfies both the above properties.

Remark 4.34. Note, choosing $A = \Omega$, we see $\boldsymbol{E}(\boldsymbol{E}_n X) = \boldsymbol{E} X$.

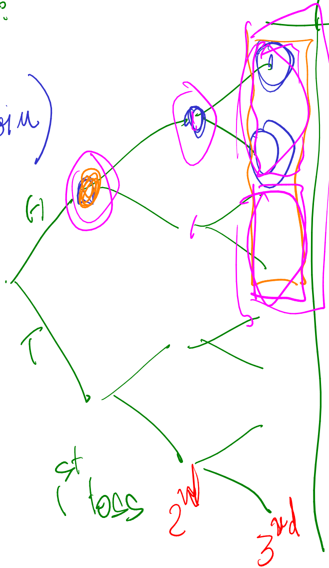
Proposition 4.35. *(1) If X, Y are two random variables and $\alpha \in \mathbb{R}$, then* $\boldsymbol{E}_n(X + \alpha Y) = \boldsymbol{E}_n X + \alpha \boldsymbol{E}_n Y$.

(2) (Tower property) *If $m \leqslant n$, then* $\boldsymbol{E}_m(\boldsymbol{E}_n X) = \boldsymbol{E}_m X$.

(3) If X is $\mathcal{F}_n$ measurable, and Y is any random variable, then $\boxed{\boldsymbol{E}_n(XY) = X \boldsymbol{E}_n Y}$.

Eg:

(Fair coin)



X	E_3X	E_2X	E_1X	$E_1(E_2X)$
1	1	1.5	2.5	2.5
2	2	1.5	2.5	
3	3	3.5	2.5	
4	4	3.5	2.5	
5	5	.		
6	6	.		
7	7	.		
8	8	.		

Proposition 4.36. *(1) If X is measurable with respect to $\mathcal{F}_n$, then $\boxed{\boldsymbol{E}_n X = X}$.*
(2) If X is independent of $\mathcal{F}_n$ then $\boxed{\boldsymbol{E}_n X = \boldsymbol{E} X}$.

Remark 4.37. We say X is independent of $\mathcal{F}_n$ if for every $\boxed{A \in \mathcal{F}_n}$ and $\boxed{B \subseteq \mathbb{R}}$, the events A and $\boxed{\{X \in B\}}$ are independent.

Example 4.38. If X only depends on the $(n+1)^{\text{th}}$, $(n+2)^{\text{th}}$, $\ldots$, n^{th} coin tosses and *not* the 1^{st}, 2^{nd}, $\ldots$, n^{th} coin tosses, then X is independent of $\mathcal{F}_n$.

$\boxed{X}$ is ind of the 1^{st} n coin tosses

$$\{X \in B\} = \{\omega \in \Omega \mid X(\omega) \in B\}$$

Proposition 4.39 (Independence lemma). *If X is independent of $\mathcal{F}_n$ and Y is $\mathcal{F}_n$-measurable, and $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a function then*

$$\boldsymbol{E}_n f(X, Y) = \sum_{i=1}^{m} f(x_i, Y)\boldsymbol{P}(X = x_i)\,, \qquad \textit{where } \{x_1, \dots, x_m\} = X(\Omega)\,.$$

$\mathcal{F}_n$ ind

$\mathcal{F}_n$ - meas

$$\boxed{E f(X) = \sum_{x_i \in \text{Range}(X)} f(x_i) \, P(X = x_i)}$$

Eg: $f(x, y) = \sqrt{X^2 + Y^2}$

X ind of $\mathcal{F}_n$

Y meas wrt $\mathcal{F}_n$

4.5. **Martingales.**

Definition 4.40. A *stochastic process* is a collection of random variables $X_0, X_1, \ldots, X_N$.

Example 4.41. Typically X_n is the wealth of an investor at time n, or S_n is the price of a stock at time n.

Definition 4.42. A stochastic process is *adapted* if X_n is $\mathcal{F}_n$-measurable for all n. (Non-anticipating.)

Remark 4.43. Requiring processes to be adapted is fundamental to Finance. Intuitively, being adapted forbids you from trading today based on tomorrows stock price. All processes we consider (prices, wealth, trading strategies) will be adapted.

Example 4.44 (Money market). Let $Y_0 = Y_0(\omega) = a \in \mathbb{R}$. Define $Y_{n+1} = (1+r)Y_n$. (Here r is the interest rate.)

Example 4.45 (Stock price). Let $S_0 \in \mathbb{R}$. Define $S_{n+1}(\omega) = \begin{cases} uS_n(\omega) & \omega_{n+1} = 1\,, \\ dS_n(\omega) & \omega_{n+1} = -1\,. \end{cases}$

$S_0 > 0$

adapted stochastic process.

Definition 4.46. We say an adapted process M_n is a martingale if $\boldsymbol{E}_n M_{n+1} = M_n$. (Recall $\boldsymbol{E}_n Y = \boldsymbol{E}(Y \mid \mathcal{F}_n)$.)

Remark 4.47. Intuition: A martingale is a "fair game".

Example 4.48 (Unbiased random walk). If $\xi_1, \ldots, \xi_N$ are i.i.d. and mean zero, then $X_n = \sum_{k=1}^n \xi_k$ is a martingale.

$X_0 = 0$

$X_{n+1} = X_n + \xi_{n+1}$

$X_1 = X_0 + \xi_1$

$X_2 = X_1 + \xi_2$

$\boxed{\text{Assume } E\xi_n = 0 \text{ \& } \xi_{n+1} \text{ is ind of } \xi_n}$

Then X_n is a mg:

$E_n X_{n+1} \overset{\text{Want}}{=} X_n$

n

$$E_n(X_{n+1}) = E_n(X_n + \xi_{n+1})$$

$$= E_n X_n + E_n \xi_{n+1}$$

$$= X_n + E\xi_{n+1} \quad (\because X_n \text{ is } \mathcal{F}_n\text{-meas}$$

$$\xi_{n+1} \text{ is } \mathcal{F}_n\text{-ind})$$

$$= X_n$$

Remark 4.49. If M is a martingale, then for every $m \leqslant n$, we must have $\boxed{\boldsymbol{E}_m M_n = M_m.}$

Remark 4.50. If M is a martingale then $\boxed{\boldsymbol{E} M_n} = \boldsymbol{E} M_0 = M_0$.

$$M_n = E_n M_{n+1}$$

$$M_{n-1} = E_{n-1} M_n = E_{n-1}\left(E_n M_{n+1} \right) \overset{\text{Tower}}{=} \boxed{E_{n-1} M_{n+1}}$$

$$M_{n-2} = E_{n-2} M_{n-1} = E_{n-2} E_{n-1} M_{n+1} = E_{n-2} M_{n+1}.$$

etc.

$$E M_n = E\left(E_0 M_n\right) \overset{mg}{=} E M_0 = M_0$$

($\because$ M_0 is $\mathcal{F}_0$ meas.

M_0 doesn't dep on any coin tosses)

4.6. **Change of measure.**

- Gambling in a Casino: If it's a martingale, then on average you won't make or lose money.
- Stock market: Bank always pays interest! Not looking for a "break even" strategy.
- Mathematical tool that helps us price securities: Find a *Risk Neutral Measure.*
  - ▷ Discounted stock price is (usually) not a martingale.
  - ▷ Invent a "risk neutral measure" which the discounted stock price is a martingale.
  - ▷ Securities can be priced by taking a conditional expectation *with respect to the risk neutral measure.* (That's the meaning of $\tilde{\boldsymbol{E}}_n$ in Proposition 4.1.)

$V_N \to$ Payoff at time N

$V_n =$ AFP at time n

$D_n = \frac{1}{(1+r)^n}$ discount factor

RNP Formula

$$\boxed{V_n = \frac{1}{D_n} \tilde{E}_n (D_N V_N)}$$

Definition 4.51. Let $D_n = (1+r)^{-n}$ be the discount factor. (So D_n\$ in the bank at time 0 becomes 1\$ in the bank at time n.)

- Invent a new probability mass function $\tilde{p}$.
- Use a tilde to distinguish between the new, invented, probability measure and the old one.
  - ▷ $\tilde{\boldsymbol{P}}$ the probability measure obtained from the PMF $\tilde{p}$ (i.e. $\tilde{\boldsymbol{P}}(A) = \sum_{\omega \in A} \tilde{p}(\omega)$).
  - ▷ $\tilde{\boldsymbol{E}}$, $\tilde{\boldsymbol{E}}_n$ conditional expectation with respect to $\tilde{\boldsymbol{P}}$ (the new "risk neutral" coin)

Definition 4.52. We say $\boldsymbol{P}$ and $\tilde{\boldsymbol{P}}$ are equivalent if for every $A \in \mathcal{F}_N$, $\boldsymbol{P}(A) = 0$ if and only if $\tilde{\boldsymbol{P}}(A) = 0$.

Definition 4.53. A *risk neutral measure* is an equivalent measure $\tilde{\boldsymbol{P}}$ under which $D_n S_n$ is a martingale. (I.e $\tilde{\boldsymbol{E}}_n(D_{n+1} S_{n+1}) = D_n S_n$.)

Remark 4.54. If there are more than one risky assets, $S^1, \dots, S^k$, then we require $D_n S_n^1, \dots, D_n S_n^k$ to all be martingales under the risk neutral measure $\tilde{\boldsymbol{P}}$.

Remark 4.55. Proposition 4.1 says that any security with payoff V_N at time N has arbitrage free price $V_n = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n(D_N V_N)$ at time n. (Called the risk neutral pricing formula.)

Proposition 4.56. *Let $\tilde{\boldsymbol{P}}$ be an equivalent measure under which the coins are i.i.d. and land heads with probability $\tilde{p}_1$ and tails with probability $\tilde{q}_1 = 1 - \tilde{p}_1$.*

(1) *Under $\tilde{\boldsymbol{P}}$, we have $\tilde{\boldsymbol{E}}_n(D_{n+1}S_{n+1}) = \frac{\tilde{p}_1 u + \tilde{q}_1 d}{1+r} D_n S_n$.*

(2) *$\tilde{\boldsymbol{P}}$ is the risk neutral measure if and only if $\tilde{p}_1 u + \tilde{q}_1 d = 1 + r$. (Explicitly $\tilde{p}_1 = \frac{1+r-d}{u-d}$, and $\tilde{q}_1 = \frac{u-(1+r)}{u-d}$.)*

(1) Compute $\tilde{E}_n(D_{n+1} S_{n+1})$:

Let $X_{n+1} = \begin{cases} u & \text{if } n+1^{\text{th}} \text{ coin is heads} \\ d & \text{if } n+1^{\text{th}} \text{ coin is tails} \end{cases}$

Note $S_{n+1} = S_n X_{n+1}$

$\Rightarrow \tilde{E}_n(D_{n+1} S_{n+1}) = D_{n+1} \tilde{E}_n(S_{n+1})$

$$= D_{n+1}\,\tilde{E}_n\left(S_n X_{n+1}\right)$$

$$= D_{n+1} S_n\,\tilde{E}_n X_{n+1} \qquad (\because S_n \text{ is } \mathcal{F}_n \text{ meas})$$

$$= \frac{D_n}{1+r} S_n\,\tilde{E} X_{n+1} \qquad (\because X_{n+1} \text{ is ind of } \mathcal{F}_n)$$

$$= D_n S_n \left(\frac{u\tilde{p}_1 + d\tilde{q}_1}{1+r}\right)$$

Theorem 4.57. *Let X_n represent the wealth of a portfolio at time n. The portfolio is self-financing portfolio if and only if the discounted wealth $D_n X_n$ is a martingale under the risk neutral measure $\tilde{\boldsymbol{P}}$.*

Remark 4.58. Recall a portfolio is *self financing* if $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$ for some $\boxed{\textit{adapted}}$ process Δ_n.

(1) That is, self-financing portfolios use only tradable assets when trading, and don't look into the future.
(2) All replication has to be done using self-financing portfolios.

Check: If X_n is self fin (Know $D_n S_n$ is a $\tilde{P}$ mg)

Then $(D_n X_n)$ is a $\tilde{P}$ - mg.

Pf: Know: $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$

Want $\tilde{E}_n(D_{n+1} X_{n+1}) = D_n X_n$

$$\tilde{E}_n\left(D_{n+1}X_{n+1}\right) = \tilde{E}_n\left(D_{n+1}\Delta_n S_{n+1} + D_{n+1}(1+r)\left(X_n - \Delta_n S_n\right)\right)$$

$$= \Delta_n \underbrace{\tilde{E}_n\left(D_{n+1}S_{n+1}\right)} + \cancel{\tilde{E}_n}\, \underbrace{D_n\left(X_n - \Delta_n S_n\right)}$$

E_n – meas.

$$= \Delta_n \cancel{D_n S_n} + D_n\left(X_n - \cancel{\Delta_n S_n}\right) = D_n X_n$$