1. Filtration & Measurability.

$$\Omega = \{N\text{-many coin toss}\}$$
$$= \{(H, H, T, \cdots, T), (T, H, \cdots, H), \cdots\}$$

<u>Def</u> Filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_{N-1} \subseteq \mathcal{F}_N$ where

$n \geq 1$: $\mathcal{F}_n = \{\text{events that only depend on the first } n\text{-coins}\}$

$n = 0$: $\mathcal{F}_0 = \{\phi, \Omega\}$

<u>Rmk</u> $\mathcal{F}_n$ = accumulated information up to time n.

<u>Eg</u> $N = 3$

$$\{\omega_1 = T\} = \cdots$$
$$= \{(T,H,H), (T,H,T), (T,T,H), (T,T,T)\}$$

- $\mathcal{F}_0 = \{\phi, \Omega\}$
- $\mathcal{F}_1 = \{\phi, \{\omega_1 = H\}, \underline{\{\omega_1 = T\}}, \Omega\}$
- $\mathcal{F}_2 = \{\phi, \{\omega_1 = H, \omega_2 = T\}, \{\omega_1 = H, \omega_2 = H\}, \{\omega_1 = T, \omega_2 = H\}, \{\omega_1 = T, \omega_2 = T\}, \Omega, \cdots\cdots\}$
- $\mathcal{F}_3 = 2^{\Omega} = \{\text{every subset of } \Omega\}$

Prop $A, B \in \mathcal{F}_n$, then $A \cup B$, $A \cap B$, $A^c \in \mathcal{F}_n$

Def A r.v $X : \Omega \to \mathbb{R}$, X is $\mathcal{F}_n$ - measurable if

$\{X \in B\} \in \mathcal{F}_n$ for all $B \subseteq \mathbb{R}$

$= \{\omega \in \Omega \mid X(\omega) \in B\}$

- X is $\mathcal{F}_n$ - measurable if X only depends on $\mathcal{F}_n$
  if X only depends on the first n-coins.

EX $X = \begin{cases} 1 & \omega_1 = H \\ -1 & \omega_1 = T \end{cases}$, $\{X \in B\} = \begin{cases} \{\omega_1 = H\} & 1 \in B, -1 \notin B \\ \{\omega_1 = T\} & -1 \in B, 1 \notin B \\ \Omega & \pm 1 \in B \\ \emptyset & 1, -1 \notin B \end{cases}$

$\therefore X$ is $\mathcal{F}_1$ - m'ble.

EX $N=100$. $X = \#$ of Heads among the first 30 coins

Q. Is X $\mathcal{F}_{30}$-m'ble? A. Yes

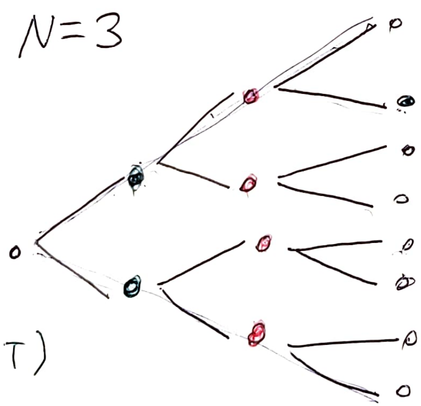
Q. Is X $\mathcal{F}_{29}$-m'ble? A. NO

Q. Is X $\mathcal{F}_{50}$-m'ble? A. Yes.

EX $N=5$. X is $\mathcal{F}_3$-m'ble

- $X(\underline{H,H,H},H,H) = X(\underline{H,H,H},T,T)$
- $X(H,H,T,H,H) \neq X(H,H,H,H,H)$

EX $N=3$



$\Pi_1(H,H,T)$

	X	Y	Z	W
(H,H,H)	1	1	1	1
(H,H,T)	1	1	1	1
(H,T,H)	2	2	1	1
(H,T,T)	2	2	1	1
(T,H,H)	3	3	1	2
(T,H,T)	3	3	2	2
(T,T,H)	4	4	2	2
(T,T,T)	5	4	2	2
	Not $\mathcal{F}_2$-m'ble	$\mathcal{F}_2$-m'ble	NOT $\mathcal{F}_1$-m'ble	$\mathcal{F}_1$-m'ble.

Prop X, Y are $\mathcal{F}_n$-m'ble, then $X+Y$, $X-Y$, XY, e^X, $\sin(X)$, $f(X,Y)$ $\mathcal{F}_n$-m'ble

2. Conditional Expectation

Ω, p : prob of H, q : prob of T $\left(\begin{matrix} p+q=1 \\ p,q>0 \end{matrix}\right)$

Def A r.v $X : \Omega \to \mathbb{R}$. A conditional Expectation of X given $\mathcal{F}_n$ $= \mathbb{E}_n[X]$ is defined via

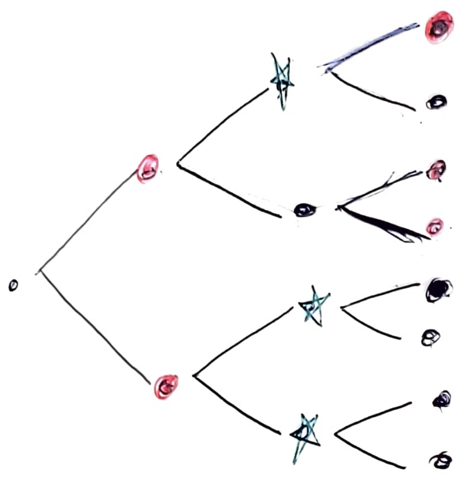
$$\mathbb{E}_n[X](\omega) = \sum_{i=1}^{m} x_i \, \mathbb{P}(X = x_i \mid \Pi_n(\omega))$$

where $\Pi_n(\omega) = \{\omega' \in \Omega \mid \omega'_1 = \omega_1, \dots, \omega'_n = \omega_n\}$ and Range$(X) = \{x_1, \dots, x_m\}$

Ex $\Pi_1(H,H,T) = \{\omega' \in \Omega \mid \omega'_1 = H\}$

Rmk $\mathbb{E}[X] = \sum_{i=1}^{m} x_i \, \mathbb{P}(X = x_i)$

<u>Ex</u> $N=3,\ p=4/5,\ q=1/5$



Y	$\mathbb{E}_2[Y]$	$\mathbb{E}_1[\mathbb{E}_2[Y]]$	$\mathbb{E}_1[Y]$
1	3/5	18/25	18/25
−1	″	″	″
2	6/5	″	″
−2	″	″	″
3	9/5	″	48/25
−3	″	48/25	″
4	12/5	48/25	″
−4	″	″	″

$$\mathbb{E}_2[Y](H,H,H) = 1\cdot \overset{4/5}{\boxed{\mathbb{P}(Y=1\mid \pi_2(H,H,H))}} - 1\cdot \overset{1/5}{\boxed{\mathbb{P}(Y=-1\mid \pi_2(H,H,H))}}$$

$$+\,2\cdot \mathbb{P}(Y=2\mid \pi_2(H,H,H)) - 2\cdot \mathbb{P}(Y=-2\mid \pi_2(H,H,H))$$

$$+\,3\cdot \mathbb{P}(Y=3\mid \pi_2(H,H,H)) - 3\,\mathbb{P}(Y=-3\mid \pi_2(H,H,H))$$

$$+\,4\cdot \mathbb{P}(Y=4\mid \pi_2(H,H,H)) - 4\cdot \mathbb{P}(Y=4\mid \pi_2(H,H,H))$$

$\mathbb{E}_2[Y](H,T,H) = 4/5 \cdot 2 + 1/5 \cdot (-2) = 6/5$

$\mathbb{E}_2[Y](T,H,H) = \frac{4}{5} \cdot 3 + \frac{1}{5} \cdot (-3) = 9/5$

$\mathbb{E}_2[Y](T,T,T) = \frac{4}{5} \cdot 4 + \frac{1}{5}(-4) = 12/5$

$$\mathbb{E}_1[\mathbb{E}_2[Y]](H,H,H) = \frac{3}{5} \cdot \left(\frac{4}{5}\right)^2 + \frac{3}{5} \cdot \left(\frac{4}{5}\right) \cdot \left(\frac{1}{5}\right) + \frac{6}{5} \cdot \left(\frac{4}{5}\right) \cdot \left(\frac{1}{5}\right)$$
$$+ \frac{6}{5} \cdot \left(\frac{1}{5}\right)^2 = \frac{18}{25}$$

$\mathbb{E}_1[Y](H,H,H) = \frac{18}{25}$

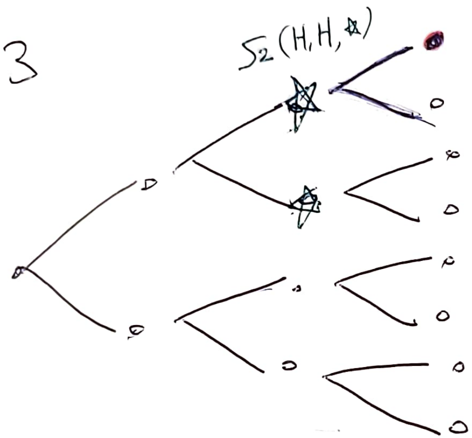
$\mathbb{E}_1[Y](T,H,H) = \frac{48}{25}.$

In our example, $\mathbb{E}_1[Y] = \mathbb{E}_1[\mathbb{E}_2[Y]]$ (Yes!)

EX Ω, p, q, S_n = stock price at time n

$$S_{n+1} \begin{cases} u \cdot S_n & \omega_{n+1} = H, \quad u > 1 \\ d \cdot S_n & \omega_{n+1} = T, \quad d < 1 \end{cases}$$

$N = 3$



$p \cdot (u \cdot S_2(H,H,*)) + q \cdot (d \cdot S_2(H,H,*))$

$''$

$p \cdot (u \cdot S_2(H,T,*)) + q \cdot (d \cdot S_2(H,T,*))$

$p \cdot (u \cdot S_2(T,H,*)) + q \cdot (d \cdot S_2(T,H,*))$

$p \cdot (u \cdot S_2 \cdot (T,T,*)) + q \cdot (d \cdot S_2(T,T,*))$

$$\mathbb{E}_2[S_3]$$

$$\mathbb{E}_2[S_3] = p \cdot u \cdot S_2 + q \cdot d \cdot S_2 = (pu + qd) \cdot S_2$$

Prop ① $\mathbb{E}_n[X]$ is $\mathcal{F}_n$-m'ble.

② $\forall A \in \mathcal{F}_n$, $\sum_{\omega \in A} \mathbb{E}_n[X](\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$

"average of $\mathbb{E}_n[X]$ over A" = "average of X over A"

$A = \Omega$

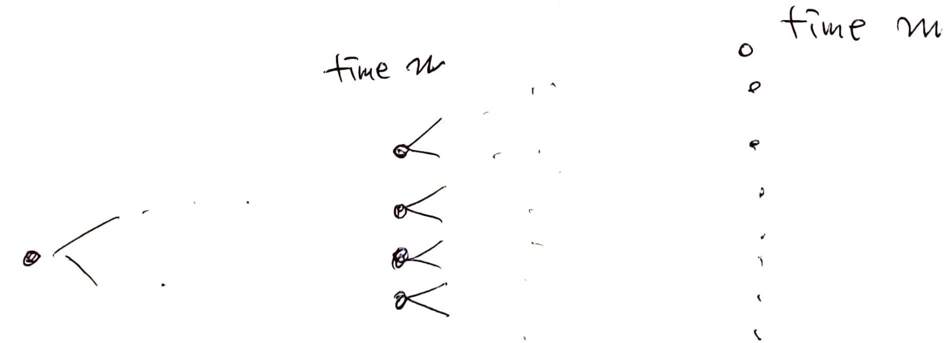
☆ A r.v Y satisfies ① Y is $\mathcal{F}_n$-m'ble

② $\forall A \in \mathcal{F}_n$, $\sum_{\omega \in A} Y(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$,

then $Y = \mathbb{E}_n[X]$.

prop 4.35 ① $\mathbb{E}_n[X + \alpha Y] = \mathbb{E}_n[X] + \alpha \cdot \mathbb{E}_n[Y]$

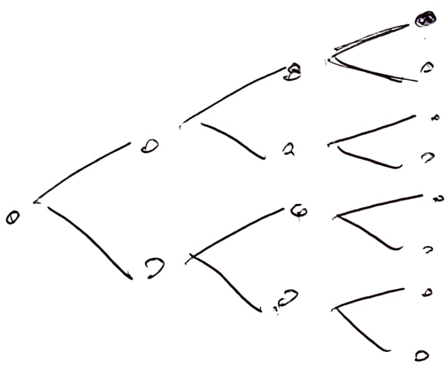
② (Tower property) $n < m$, $\mathbb{E}_n[\mathbb{E}_m[X]] = \mathbb{E}_n[X]$.



③ ("Taking out what's known") X is $\mathcal{F}_n$-m'ble,

$$\mathbb{E}_n[XY] = X \cdot \mathbb{E}_n[Y]$$

<u>EX</u> $N=3$, $p=\frac{4}{5}$, $q=\frac{1}{5}$



X	Y	$\mathbb{E}_2[Y]$	$\mathbb{E}_2[XY]$
100	1	3/5	
100	-1	"	
0	2	6/5	
0	-2	"	
5	3	9/5	
5	-3	"	
-3	4	12/5	
-3	-4	"	

X is $\mathcal{F}_2$-m'ble. WTS $\mathbb{E}_2[XY] = X \cdot \mathbb{E}_2[Y]$

$$\mathbb{E}_2[XY](H,H,H) = 100 \cdot \underbrace{1}_{a} \cdot \frac{4}{5} + 100 \cdot \underbrace{(-1)}_{b} \cdot \frac{1}{5} = 100 \underbrace{\left(1 \cdot \frac{4}{5} + (-1) \cdot \frac{1}{5}\right)}_{\mathbb{E}_2[Y]}$$

$$\mathbb{E}_2[XY] = X \cdot \mathbb{E}_2[Y].$$

<u>prop 4.36</u> (1) X is $\mathcal{F}_n$-m'ble, then $\mathbb{E}_n[X] = X$

<u>pf</u> $\mathbb{E}_n[X] = \mathbb{E}_n[X \cdot 1] = X \cdot \mathbb{E}_n[1] = X$.

(2) X is independent of $\mathcal{F}_n$, then $\mathbb{E}_n[X] = \mathbb{E}[X]$.

<u>pf</u> $Y = \mathbb{E}[X]$. WTS $\mathbb{E}_n[X] = Y$

① Y is $\mathcal{F}_n$-m'ble.

② $\forall A \in \mathcal{F}_n$, $\displaystyle\sum_{\omega \in A} Y(\omega) P(\omega) = \sum_{\omega \in A} \mathbb{E}[X] \cdot P(\omega)$

$$= \mathbb{E}[X] \cdot \sum_{\omega \in A} P(\omega) = \mathbb{E}[X] \cdot \mathbb{P}(A)$$

(Let's say Range of $X = \{x_1, \cdots, x_m\}$)

$$= \left(\sum_{i=1}^{m} x_i \mathbb{P}(X = x_i) \right) \cdot \mathbb{P}(A)$$

$$= \sum_{i=1}^{m} x_i \mathbb{P}(\{X = x_i\} \cap A)$$

$$= \sum_{i=1}^{m} x_i \sum_{\omega \in A \cap \{X = x_i\}} \mathbb{P}(\omega) = \sum_{i=1}^{m} \sum_{\omega \in A \cap \{X = x_i\}} x_i \mathbb{P}(\omega)$$

$$= \sum_{i=1}^{m} \sum_{\omega \in A \cap (X = x_i)} X(\omega) \cdot \mathbb{P}(\omega) = \sum_{\omega \in A} X(\omega) P(\omega). \qquad \blacksquare$$

prop (Ind Lem). X is independent of $\mathcal{F}_n$, Y is $\mathcal{F}_n$-m'ble

$f : \mathbb{R}^2 \to \mathbb{R}$.

$$\mathbb{E}_n[f(X,Y)] = \underline{\sum_{i=1}^{m} f(x_i, Y)\, \mathbb{P}(X = x_i)} \quad Z$$

Heuristic $Y = y$

$$\mathbb{E}_n[f(X,Y)] = \mathbb{E}_n[f(X,y)] = \mathbb{E}[f(X,y)]$$

$$= \sum_{i=1}^{m} f(x_i, y)\, \mathbb{P}(X = x_i)$$

pf ① Z is $\mathcal{F}_n$-m'ble.

② $\forall A \in \mathcal{F}_n$, $\displaystyle\sum_{\omega \in A} Z(\omega) p(\omega) = \sum_{\omega \in A} f(X,Y)(\omega) P(\omega)$

$$\sum_{\omega \in A} Z(\omega)\, p(\omega) = \sum_{\omega \in A} \left[\sum_{i=1}^{m} f(x_i, Y)(\omega)\, \mathbb{P}(X = x_i) \right] p(\omega)$$

$\cancel{\phi}$

Range$(Y) = \{y_1, \dots, y_k\}$

$$= \sum_{j=1}^{k} \sum_{\omega \in A \cap (Y = y_j)} \left[\sum_{i=1}^{m} f(x_i, y_j)\, \mathbb{P}(X = x_i) \right] \mathbb{P}(\omega)$$

$$= \sum_{j=1}^{k} \left[\sum_{i=1}^{m} f(x_i, y_j)\, \mathbb{P}(X = x_i) \right] \mathbb{P}(A \cap (Y = y_j))$$

$$= \sum_{j=1}^{k} \left[\sum_{i=1}^{m} f(x_i, y_j) \right] \mathbb{P}\big(A \cap (X = x_i) \cap (Y = y_j)\big)$$

$\star$

$$= \sum_{\omega \in A} f(X, Y)(\omega) \cdot \mathbb{P}(\overset{\omega}{\cancel{A}})$$

$\blacksquare$

$$A = \bigsqcup_{i=1}^{m}$$

$$A = \bigcup_{i=1}^{m} \bigcup_{j=1}^{k} A \cap (X = x_i) \cap (Y = y_j)$$

$$\circledast = \sum_{j=1}^{k} \sum_{i=1}^{m} f(x_i, y_j)\, \mathbb{P}\left(A \cap (X = x_i) \cap (Y = y_j)\right)$$

$$= \sum_{j=1}^{k} \sum_{i=1}^{m} f(x_i, y_j) \left[\sum_{\omega \in A \cap (X = x_i) \cap (Y = y_j)} \mathbb{P}(\omega) \right]$$

$$= \sum_{j=1}^{k} \sum_{i=1}^{m} \sum_{\omega \in A \cap (X = x_i) \cap (Y = y_j)} f(x_i, y_j)\, \mathbb{P}(\omega)$$

$$= \sum_{j=1}^{k} \sum_{i=1}^{m} \sum_{\omega \in A \cap (X = x_i) \cap (Y = y_j)} f(X(\omega), Y(\omega))\, \mathbb{P}(\omega)$$

$$= \sum_{\omega \in A} f(X, Y)(\omega)\, \mathbb{P}(\omega).$$

$\square$