2. Syllabus Overview

- Class website and full syllabus: https://www.math.cmu.edu/~gautam/sj/teaching/2022-23/944-scalc-fina
- TA's: Jonghwa Park <jonghwap@andrew.cmu.edu>.
- Homework Due: 10:10AM Oct 27, Nov 3, 10, 22, 29, Dec 6
- Midterm: Tue, Nov 15, in class
- **Homework:**
  - ▷ Good quality scans please! Use a scanning app, and not simply take photos. (I use Adobe Scan.)
  - ▷ 20% penalty if turned in within an hour of the deadline. 100% penalty after that.
  - ▷ One homework assignments can be turned in 24h late without penalty.
  - ▷ Bottom homework score is dropped from your grade (personal emergencies, interviews, other deadlines, etc.).
  - ▷ Collaboration is encouraged. Homework is not a test – ensure you learn from doing the homework.
  - ▷ You must write solutions independently, and can only turn in solutions you fully understand.
- **Academic Integrity**
  - ▷ Zero tolerance for violations (automatic **R**).
  - ▷ Violations include:
    - Not writing up solutions independently and/or plagiarizing solutions
    - Turning in solutions you do not understand.
    - Seeking, receiving or providing assistance during an exam.
  - ▷ All violations will be reported to the university, and they may impose additional penalties.
- **Grading:** 10% homework, 30% midterm, 60% final. + attendance

Course Outline.

- Review of Fundamentals: Replication, arbitrage free pricing.
- Quick study of the multi-period binomial model.
  - ▷ Simple example of replication / arbitrage free pricing.
  - ▷ Understand conditional expectations. (Have an explicit formula.)
  - ▷ Understand measurablity / adaptedness. (Can be stated easily in terms of coin tosses that have / have not occurred.)
  - ▷ Understand risk neutral measures. Explicit formula!
- Develop tools to price securities in continuous time.
  - ▷ Brownian motion (not as easy as coin tosses)
  - ▷ Conditional expectation: No explicit formula!
  - ▷ Itô formula: main tool used for computation. Develop some intuition.
  - ▷ Measurablity / risk neutral measures: much more abstract. Complete description is technical. But we need a working knowledge.
  - ▷ Derive and understand the Black-Scholes formula.

3. Replication and Arbitrage

3.1. Replication and arbitrage free pricing.

- Start with a *financial market* consisting of traded assets (stocks, bonds, money market, options, etc.)
- We model the price of these assets through random variables (stochastic processes).
- **No Arbitrage Assumption:**
  - ▷ In order to make money, you have to take risk. (Can't make something out of nothing.)
  - ▷ Mathematically: For any trading strategy such that $X_0 = 0$, and $X_n \geqslant 0$, you must also have $X_n = 0$ almost surely.
  - ▷ Equivalently: There doesn't exist a trading strategy with $X_0 = 0$, $X_n \geqslant 0$ and $\boldsymbol{P}(X_n > 0) > 0$.

Stock

Bank

Market

X_0 = Wealth at time $0 = 0$

& X_n = " " " $n \geq 0$

then $X_n = 0$

- **Arbitrage free price**
  - ▷ Now consider a non-traded asset Y (e.g. an option). How do you price it?
  - ▷ *Arbitrage free price:* If given the opportunity to trade Y at price V_0, the market remains arbitrage free, then we say V_0 is the arbitrage free price of Y.

No options.

Stocks

Bank

Market (A. F.)

Want to trade a call option?

Call option at price $\boxed{V_0}$ → If market remains A F

then V_0 = A F P

- **Replication**
  - *We will almost always find the arbitrage free price by* ***replication.***
  - Say the non-traded asset pays V_N at time N (e.g. call options).
  - Try and *replicate the payoff*:
    - Start with X_0 dollars.
    - Use only traded assets and ensure that at maturity $X_N = V_N$.
  - Then the arbitrage free price is uniquely determined, and must be X_0.

Remark 3.1. The arbitrage free price is *unique* if and only if there is a replicating strategy! In this case, the arbitrage free price is exactly the initial capital of the replicating strategy.

(FTAP).

3.2. **Example: One period Binomial model.**

- Consider a market with a stock, and money market account.
- Interest rate for borrowing and lending is r. No transaction costs. Can buy and sell fractional quantities of the stock.
- *Model assumption:* Flip a coin that lands heads with probability $p_1 \in (0,1)$ and tails with probability $q_1 = 1 - p_1$. Model $S_1 = uS_0$ if heads, and $S_1 = dS_0$ if tails.
  - ▷ S_0 is stock price at time 0 (known).
  - ▷ S_1 is stock price after one time period (random).
  - ▷ u, d are model parameters (pre-supposed). Called the up and down factors. (Will always assume $0 < d < u$.)

Proposition 3.2. *There's no arbitrage in this model if and only if* $\boxed{d < 1 + r < u.}$

Proof.

Heads: $S_1 = u S_0$

S_0

Tails: $S_1 = d S_0$

(Price at time 0)

Check : Say $\boxed{d = 1+r}$

Find arb :

At time 0 $\begin{cases} 1 \text{ share Stock} \\ -S_0 \quad \text{cash} \end{cases}$

$$X_0 = 0 = \underbrace{S_0}_{1 \text{ share}} - \underbrace{S_0}_{\text{cash.}}$$

$$X_1 = \underset{\text{new stock price}}{\underline{S_1}} - (1+r)S_0$$

$$\boxed{(u-(1+r))S_0} > 0$$

$$= \begin{cases} \boxed{uS_0 - (1+r)S_0} & \text{if heads} \\ \boxed{dS_0 - (1+r)S_0} & \text{if tails} \end{cases}$$

$$< 0$$

Proposition 3.3. *Say a security pays $\boxed{V_1}$ at time 1 (V_1 can depend on whether the coin flip is heads or tails). The arbitrage free price at time 0 is given by*

$$\boxed{V_0 = \frac{1}{1+r}\big(\tilde{p}_1 V_1(H) + \tilde{q}_1 V_1(T)\big) = \boxed{\frac{1}{1+r}\tilde{\boldsymbol{E}} V_1}}\,, \quad \text{where } \boxed{\tilde{p}_1 = \frac{1+r-d}{u-d}\,, \quad \tilde{q}_1 = \frac{u-(1+r)}{u-d}}\,.$$

The replicating strategy holds $\Delta_0 = \dfrac{V_1(H) - V_1(T)}{(u-d)S_0}$ *shares of stock at time 0.*

Proof.

Replicate the payoff by buying $\boxed{\Delta_0}$ shares at time 0

& starting with X_0 \$

$$X_0 = \underbrace{\Delta_0 S_0}_{\text{Stock}} + \underbrace{(X_0 - \Delta_0 S_0)}_{\text{cash}}$$

$$X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) \overset{\text{Want}}{=} V_1$$

If heads: $X_1 = \Delta_0 u S_0 + (1+r)(X_0 - \Delta_0 S_0) = V_1(H)$ $\times \tilde{p}$

If tails: $X_1 = \Delta_0 d S_0 + (1+r)(X_0 - \Delta_0 S_0) = V_1(T)$ $\times \tilde{q}$

2 Eq & 2 unknowns $(X_0 \,\&\, \Delta_0)$ Solve.

Find $\tilde{p}$ & $\tilde{q}$ so that $\boxed{\tilde{p} + \tilde{q} = 1 \;\;\&\;\; u\tilde{p} + d\tilde{q} = (1+r)}$

$$\tilde{p}\, V_1(H) + \tilde{q}\, V_1(T) = \Delta_0 S_0 \underbrace{(\tilde{p} u + \tilde{q} d)}_{1+r} + \underbrace{(\tilde{p} + \tilde{q})}_{1} (1+r)(X_0 - \Delta_0 S_0)$$

$$= (1+r)\cancel{\Delta_0 S_0} + (1+r)(X_0 - \cancel{\Delta_0 S_0})$$

$$\boxed{\Rightarrow X_0 = \frac{\tilde{p}\, V_1(H) + \tilde{q}\, V_1(T)}{(1+r)}}$$

Find Δ_0: Subtract the 2 eq $\Delta_0 = \dfrac{V_1(H) - V_1(T)}{(u-d)\, S_0}$

Find $\tilde{p}, \tilde{q}$: $\tilde{p} = 1 - \tilde{q}$

$$u\tilde{p} + d\tilde{q} = 1+r \Rightarrow u\tilde{p} + d(1-\tilde{p}) = 1+r$$

$$\Rightarrow \tilde{p}(u-d) = 1+r-d$$

$$\Rightarrow \tilde{p} = \frac{1+r-d}{u-d} \quad \& \quad \tilde{q} = \frac{u-(1+r)}{d}$$

4. **Multi-Period Binomial Model.**

- Same setup as the one period case $0 < d < 1 + r < u$, and toss coins that land heads with probability p_1 and tails with probability q_1.
- Except now the security matures at time $N > 1$.
- Stock price: $S_{n+1} = uS_n$ if $n + 1$-th coin toss is heads, and $S_{n+1} = dS_n$ otherwise.
- To replicate it a security, we start with capital X_0.
- Buy Δ_0 shares of stock, and put the rest in cash.
- Get $X_1 = \Delta_0 S_1 + (1 + r)(X_0 - \Delta_0 S_0)$.
- Repeat. *Self Financing Condition:* $X_{n+1} = \Delta_n S_{n+1} + (1 + r)(X_n - \Delta_n S_n)$.
- *Adaptedness:* Δ_n can only depend on outcomes of coin tosses before n!

Proposition 4.1. *Consider a security that pays V_N at time N. Then for any $n \leqslant N$:*

$$\boxed{V_n = \frac{1}{(1+r)^{N-n}} \tilde{\boldsymbol{E}}_n V_N}\,, \qquad \boxed{\Delta_n = \frac{V_{n+1}(\omega_{n+1} = H) - V_{n+1}(\omega_{n+1} = T)}{(u-d)S_n}}\,.$$

- *V_n is the arbitrage free price at time $n \leqslant N$.*
- *Δ_n is the number of shares held in the replicating portfolio at time n (trading strategy).*

Question 4.2. *Why does this work?*

Question 4.3. *What is $\tilde{\boldsymbol{E}}_n$? (It's different from $\boldsymbol{E}$, and different from $\boldsymbol{E}_n$).*

$\tilde{E}_n$ → $\sim$: Exp wrt the "Risk Neutral Measure"

→ n → "Conditional exp on $\mathcal{F}_n$" (IOU)

4.1. **Quick review probability (finite Sample spaces).** This is just a quick review for you to fix notation. You should already be familiar with this material from previous courses, and we won't go over it in class. We will, however, spend some time studying conditional expectation.

Let $N \in \mathbb{N}$ be large (typically the maturity time of financial securities).

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Definition 4.4. The $\boxed{\textit{sample space}}$ is the set $\Omega = \{(\omega_1, \ldots, \omega_N) \mid \text{each } \omega_i \text{ represents the outcome of a coin toss}\}$.

▷ E.g. $\omega_i \in \{H, T\}$, or $\omega_i \in \{\pm 1\}$. (Each ω_i could also represent the outcome of the roll of a M sided die.)

Definition 4.5. A *sample point* is a point $\omega = (\omega_1, \ldots, \omega_N) \in \Omega$.

▷ Each sample point represents the outcome of a sequence of *all* coin tosses from 1 to N.

Definition 4.6. A *probability mass function* (PMF for short) is a function $p \colon \Omega \to [0,1]$ such that $\sum_{\omega \in \Omega} p(\omega) = 1$.

Example 4.7. Typical example: Fix $p_1 \in (0,1)$, $q_1 = 1 - p_1$ and set $p(\omega) = p_1^{H(\omega)} q_1^{T(\omega)}$. Here $H(\omega)$ is the number of heads in the sequence $\omega = (\omega_1, \ldots, \omega_N)$, and $T(\omega)$ is the number of tails.

Definition 4.8. An event is a subset of Ω. Define $\boldsymbol{P}(A) = \sum_{\omega \in A} p(\omega)$.

▷ $\boldsymbol{P}$ is called the probability measure associated with the PMF p.

Example 4.9. $A\{\omega \in \Omega \mid \omega_1 = +1\}$. Check $\boldsymbol{P}(A) = p_1$.

4.2. **Random Variables and Independence.**

Definition 4.10. A *random variable* is a function $X \colon \Omega \to \mathbb{R}$.

Example 4.11. $X(\omega) = \begin{cases} 1 & \omega_2 = +1\,, \\ -1 & \omega_2 = -1\,, \end{cases}$ is a random variable corresponding to the outcome of the second coin toss.

Definition 4.12. The *expectation* of a random variable X is $\boldsymbol{E}X = \sum X(\omega)p(\omega)$.

Remark 4.13. Note if $\operatorname{Range}(X) = \{x_1, \dots, x_n\}$, then $\boldsymbol{E}X = \sum X(\omega)p(\omega) = \sum_1^n x_i \boldsymbol{P}(X = x_i)$.

Definition 4.14. The *variance* of a random variable is $\operatorname{Var}(X) = \boldsymbol{E}(X - \boldsymbol{E}X)^2$.

Remark 4.15. Note $\operatorname{Var}(X) = \boldsymbol{E}X^2 - (\boldsymbol{E}X)^2$.

Definition 4.16. Two events are independent if $\boldsymbol{P}(A \cap B) = \boldsymbol{P}(A)\boldsymbol{P}(B)$.

Definition 4.17. The events $A_1, \dots, A_n$ are independent if for any sub-collection $A_{i_1}, \dots, A_{i_k}$ we have

$$\boldsymbol{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \boldsymbol{P}(A_{i_1})\boldsymbol{P}(A_{i_2}) \cdots \boldsymbol{P}(A_{i_k})\,.$$

Remark 4.18. When $n > 2$, it is not enough to only require $\boldsymbol{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \boldsymbol{P}(A_1)\boldsymbol{P}(A_2) \cdots \boldsymbol{P}(A_n)$

Definition 4.19. Two random variables are independent if $\boldsymbol{P}(X = x,\ Y = y) = \boldsymbol{P}(X = x)\boldsymbol{P}(Y = y)$ for all $x, y \in \mathbb{R}$.

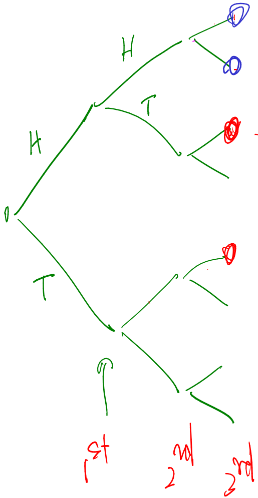
Definition 4.20. The random variables $X_1, \dots, X_n$ are independent if for all $x_1, \dots, x_n \in \mathbb{R}$ we have

$$\boldsymbol{P}(X_1 = x_1,\ X_2 = x_2, \dots, X_n = x_n) = \boldsymbol{P}(X_1 = x_1)\boldsymbol{P}(X_2 = x_2) \cdots \boldsymbol{P}(X_n = x_n)\,.$$

Remark 4.21. Independent random variables are uncorrelated, but not vice versa.

4.3. **Filtrations.**

- Let $\Omega = \{(\omega_1, \dots, \omega_N) \mid \text{each } \omega_i \in \pm 1 \text{ represents the outcome of a coin toss.}\}$.
  - ▷ It is convenient to visualize Ω and random variables by drawing trees.
  - ▷ E.g. X = outcome (± 1) of the second coin toss, Y = number of heads, etc.



	X	Y
	1	3
	1	2
(H, T, H)	−1	2
	−1	1
(T, H, H)	1	2
	1	1
	−1	1
	−1	0

$$X = \begin{cases} 1 & 2^{nd} \text{ toss} = H \\ -1 & 2^{nd} \text{ toss} = T \end{cases}$$

X is $\mathcal{F}_1$ - meas NO

X is $\mathcal{F}_2$ - meas Yes

X is $\mathcal{F}_2$ meas Yes

Y is $\mathcal{F}_3$ meas.

Definition 4.22. We define a *filtration* on Ω as follows:

- ▷ $\mathcal{F}_0 = \{\emptyset, \Omega\}$.
- ▷ $\mathcal{F}_1$ = all events that can be described by only the first coin toss. E.g. $A = \{\omega \mid \omega_1 = +1\} \in \mathcal{F}_1$.
- ▷ $\mathcal{F}_2$ = all events that can be described by only the first two coin toss.
  - E.g. $A = \{\omega \mid \omega_1 = +1\} \in \mathcal{F}_2$, $B = \{\omega \mid \omega_1 = +1, \omega_2 = -1\} \in \mathcal{F}_2$.
- ▷ $\mathcal{F}_n$ = all events that can be described by only the first n coin tosses.
  - E.g. $A = \{\omega \mid \omega_1 = 1, \omega_3 = -1, \omega_n = 1\} \in \mathcal{F}_n$.

Remark 4.23. Note $\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_N = \mathcal{P}(\Omega)$.

Remark 4.24. If $A, B \in \mathcal{F}_n$, then so do A^c, B^c, $A \cap B$, $A \cup B$, $A - B$, $B - A$.

$$F_1 = \{\emptyset, \Omega, \{(H,H,H), (H,H,T), (H,T,H), (H,T,T)\}, \ldots\}$$

Definition 4.25. Let $n \in \{0, \ldots, N\}$. We say a random variable X is $\boxed{\mathcal{F}_n\text{-measurable}}$ if $\boxed{X(\omega)}$ only depends on $\omega_1, \ldots, \omega_n$. (and not $\omega_{n+1}, \omega_{n+2}, \ldots, \omega_N$)

▷ Equivalently, for any $B \subseteq \mathbb{R}$, the event $\{X \in B\} \in \mathcal{F}_n$.

Remark 4.26 (Use in Finance). For every n, the trading strategy at time n (denoted by Δ_n) must be $\mathcal{F}_n$ measurable. We can not trade today based on tomorrows price.

Example 4.27. If we represent Ω as a tree, $\mathcal{F}_n$ measurablity can be visualized by checking constancy on leaves.

4.4. Conditional expectation.

Definition 4.28. Let X be a random variable, and $n \leqslant N$. We define $\boldsymbol{E}(X \mid \mathcal{F}_n) = \boldsymbol{E}_n X$ to be the *random variable* given by

$$\boldsymbol{E}_n X(\omega) = \sum_{x_i \in \mathrm{Range}(X)} x_i \boldsymbol{P}(X = x_i \mid \Pi_n(\omega)) \quad \longleftarrow \mathcal{F}_n\text{- meas.}$$

$$\text{where} \qquad \Pi_n(\omega) = \{\omega' \in \Omega \mid \omega'_1 = \omega_1, \ \ldots, \omega'_n = \omega_n\} \in \mathcal{F}_n$$

Remark 4.29. The above formula does not generalize well to infinite probability spaces. We will develop certain properties of $\boldsymbol{E}_n$, and then only use those properties going forward.

Example 4.30. If we represent Ω as a tree, $\boldsymbol{E}_n X$ can be computed by averaging over leaves.

Remark 4.31. $\boxed{\boldsymbol{E}_n X}$ is the "best approximation" of X given only the first n coin tosses.

$$EX = \sum x_i P(X = x_i) \qquad\qquad P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Proposition 4.32. *The conditional expectation* $\boxed{\boldsymbol{E}_n X}$ *defined by the above formula satisfies the following two properties:*

(1) $\boldsymbol{E}_n X$ *is an* $\mathcal{F}_n$*-measurable* random variable.

(2) *For every* $A \in \mathcal{F}_n$, $\boxed{\sum_{\omega \in A} \boldsymbol{E}_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega).}$

Remark 4.33. This property is used to define conditional expectations in the continuous time setting. It turns out that there is exactly one random variable that satisfies both the above properties; and thus we define $\boldsymbol{E}_n X$ to be the unique random variable which satisfies both the above properties.

Remark 4.34. Note, choosing $A = \Omega$, we see $\boldsymbol{E}(\boldsymbol{E}_n X) = \boldsymbol{E} X$.

$\boxed{A \in \mathcal{F}_n,}$

Avg over A of $X = \frac{1}{P(A)} \boxed{\sum_{\omega \in A} p(\omega) X(\omega)}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$

Avg " A of $E_n X = \frac{1}{P(A)} \boxed{\sum_{\omega \in A} p(\omega) E_n X(\omega)}$

Proposition 4.35. *(1) If X, Y are two random variables and $\alpha \in \mathbb{R}$, then* $\boldsymbol{E}_n(X + \alpha Y) = \boldsymbol{E}_n X + \alpha \boldsymbol{E}_n Y$.
(2) (Tower property) *If $m \leqslant n$, then* $\boldsymbol{E}_m(\boldsymbol{E}_n X) = \boldsymbol{E}_m X$.
(3) If X is $\mathcal{F}_n$ measurable, and Y is any random variable, then $\boldsymbol{E}_n(XY) = X \boldsymbol{E}_n Y$.

$X \rightarrow \mathcal{F}_n$ - meas,

$E_n X = X$

Proposition 4.36. *(1) If X is measurable with respect to $\mathcal{F}_n$, then $\boldsymbol{E}_n X = X$.*
(2) If X is independent of $\mathcal{F}_n$ then $\boldsymbol{E}_n X = \boldsymbol{E} X$.

Remark 4.37. We say X is independent of $\mathcal{F}_n$ if for every $A \in \mathcal{F}_n$ and $B \subseteq \mathbb{R}$, the events A and $\{X \in B\}$ are independent.

Example 4.38. If X only depends on the $(n+1)^{\text{th}}$, $(n+2)^{\text{th}}$, $\dots$, n^{th} coin tosses and *not* the 1^{st}, 2^{nd}, $\dots$, n^{th} coin tosses, then X is independent of $\mathcal{F}_n$.

Proposition 4.39 (Independence lemma)**.** *If X is independent of $\mathcal{F}_n$ and Y is $\mathcal{F}_n$-measurable, and $f\colon \mathbb{R} \to \mathbb{R}$ is a function then*

$$\boldsymbol{E}_n f(X, Y) = \sum_{i=1}^{m} f(x_i, Y) \boldsymbol{P}(X = x_i)\,, \qquad \text{where } \{x_1, \dots, x_m\} = X(\Omega)\,.$$