# Stochastic Calculus for Finance I: Final.

2022-12-08, Pittsburgh

- *This is a closed book test. You may use a calculator. You may not give or receive assistance.*
- *Your calculator must not be able to access the internet, or store/read document files (PDF, word, etc.)*
- *You have 3 hours. The exam has a total of 7 questions and 35 points.*
- *The questions are roughly ordered by difficulty. Good luck.*

In this exam $W$ always denotes a standard Brownian motion, and the filtration $\{\mathcal{F}_t \mid t \geqslant 0\}$ (if not otherwise specified) is the Brownian filtration.

$\boxed{5}$ 1. Given $\alpha, \beta \in \mathbb{R}$ define $X_t = W_t^4 + \alpha t W_t^2 + \beta t^2$. Find all $\alpha, \beta \in \mathbb{R}$ for which $X$ is a martingale.

$\boxed{5}$ 2. Let $\alpha \in \mathbb{R}$, and $0 \leqslant s < t$. Compute $\boldsymbol{E}_s \mathbf{1}_{\{W_t > \alpha\}}$. (Recall $\mathbf{1}_{\{W_t > \alpha\}}$ is a random variable which is 1 if $W_t > \alpha$ and 0 otherwise.) Express your final answer in terms of $\alpha$, $W_t$, $W_s$ and the CDF of the standard normal without using expectations, conditional expectations or integrals.

$\boxed{5}$ 3. Let $c(t, x)$ denote the price of a European call given by the Black-Scholes formula. Recall that $c$ also depends on the maturity time $T$, strike price $K$, interest rate $r$ and volatility $\sigma$. Compute

$$\lim_{K \to 0} c(t, x) \qquad \text{and} \qquad \lim_{T \to \infty} c(t, x)\,.$$

$\boxed{5}$ 4. Let $W$ and $B$ be two standard Brownian motions which are independent of each other. Let $X_t = W_t e^{-3tB_t}$.

(a) Find the quadratic variation of $X$.

(b) Find the joint quadratic variation between $X$ and $B$.

Express your answer to both parts in the form

$$\int_0^t f(s, W_s, B_s)\, ds + \int_0^t g(s, W_s, B_s)\, dW_s + \int_0^t h(s, W_s, B_s)\, dB_s\,,$$

for some functions $f$, $g$, and $h$ that you compute explicitly.

$\boxed{5}$ 5. Consider a market consisting of a money market account and a stock. The continuously compounded interest rate in the money market account is $r \geqslant 0$, and the price of the stock is modelled by a geometric Brownian motion with mean return rate $\alpha$ and volatility $\sigma$. Consider an European put option with strike $K$, maturity $T$. Let $p(t, x)$ denote the price of this at time $t \leqslant T$ when the spot price of the stock is $x$. For $0 \leqslant s \leqslant t \leqslant T$, compute $\tilde{\boldsymbol{E}}_s p(t, S_t)$. Here $\tilde{\boldsymbol{E}}_s$ denotes conditional expectation (at time $s$) with respect to the risk neutral measure. Express your final answer in terms of $p$, $s$, $t$, $T$, $S_s$, $S_t$ and the model parameters $r, \sigma, \alpha$, without involving integrals, expectations or conditional expectations. (Your final answer may not involve derivatives of $p$; but may involve special functions such as the CDF of the standard normal.)

$\boxed{5}$ 6. Consider a discrete time market consisting of a bank and a stock. Let $S_n$ denote the stock price at time $n$, and we know $S_0 = \$100$. The stock price changes according to the flip of a biased coin that lands heads with probability 1% and tails with probability 99%. If the coin lands heads the stock price increases by 5% (i.e. $S_{n+1} = 1.05 S_n$), and if the coin lands tails the stock price decreases by 1% (i.e. $S_{n+1} = 0.99 S_n$). The interest rate $r = 1\%$. Consider a security that matures at time $N = 6$. At maturity, the security pays one share of the stock if at least one coin flip was heads. If all coin flips were tails, the security pays nothing. Find the arbitrage free price of this security at time 0. Also find the number of shares in the replicating portfolio at time 0. Round your final answers to two decimal places. (I recommend rounding the answer of intermediate steps to three decimal places.)

$\boxed{5}$ 7. Consider a market consisting of a money market account and a stock. We know that up to time $T_1$ the interest rate is $r_1$. After time $T_1$, the interest rate becomes $r_2$. (Both $r_1$ and $r_2$ are fixed, non-random, constants.) The price of the stock is modelled by a geometric Brownian motion with mean return rate $\alpha$ and volatility $\sigma$. Consider an European call option with strike $K$, maturity $T > T_1$. Find the arbitrage free price of this option at time $t < T_1$. Express your final answer in terms of $t$, $T_1$, $T$, the CDF of the standard normal, the stock price, and the model parameters $r_1$, $r_2$, $\alpha$, $\sigma$ without using expectations, conditional expectations or integrals.