21-720 Measure Theory.

2022-10-08

- *This is a closed book test. You may not use phones, calculators, or other electronic devices.*
- *You may not give or receive assistance.*
- *You have 90 minutes. The exam has a total of 4 questions and 40 points.*
- *You may use any result from class or homework* ***PROVIDED*** *it is* independent *of the problem you want to use the result in. (You must also* ***CLEARLY*** *state the result you are using.)*

In this exam $\mathcal{L}(\mathbb{R}^d)$ denotes the Lebesgue σ-algebra on $\mathbb{R}^d$, $\mathcal{B}(X)$ denotes the Borel σ-algebra on a metric space X, and λ denotes the Lebesgue measure on $\mathbb{R}^d$.

[10] 1. Let $A \in \mathcal{L}(\mathbb{R})$ be such that $\lambda(A) < \infty$. Must $\lim_{n\to\infty} \lambda(A \cap (A + n))$ exist? Prove your answer.

[10] 2. Let $A \in \mathcal{B}(\mathbb{R}^2)$, and $B = \{x \in \mathbb{R} \mid (x, 0) \in A\} \subseteq \mathbb{R}$. Must $B \in \mathcal{B}(\mathbb{R})$? If yes, prove it. If no, find a counter example. [This is a special case of a result that was proved in your homework. Please provide a complete proof here, that is independent of the corresponding homework problem.]

[10] 3. Let X be a set, Σ be a σ-algebra on X, and μ be a positive measure on (X, Σ). Suppose $f_n \colon X \to \mathbb{R}$ is a sequence of integrable functions which converge pointwise almost everywhere to a function f. Suppose further there exist $a, b \in (0, \infty)$ such that $a \leqslant \int_X |f_n| \, d\mu \leqslant b$ for all $n \in \mathbb{N}$.

   (a) True or false: f is integrable, and $\left| \int_X f \, d\mu \right| \leqslant b$? If true, prove it. If false, find a counter example.

   (b) True or false: $\int_X |f| \, d\mu \geqslant a$? If true, prove it. If false, find a counter example.

[10] 4. True or false:

   For every $A \subseteq \mathbb{R}^d$ with $\lambda^*(A) < \infty$ we have $\lim_{n\to\infty} \lambda^*(A \cap B(0, n)) = \lambda^*(A)$.

   Prove your answer. [Note: A need not be Lebesgue measurable. Here λ^* denotes the Lebesgue outer measure.]