LECTURE NOTES ON MEASURE THEORY
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Contents

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1. Preface.

These notes originated as slides I used while teaching this course remotely in 2020. They mainly contain theorem statements and definitions. I projected them (spaced out) in class, and filled in the proofs by writing over them with a tablet. The annotated version of these slides with handwritten proofs, can be found on the 2020 website. The LaTeXsource of these slides is also available on git.

2. Sigma Algebras and Measures

- **Motivation:** Suppose $f_n \colon [0,1] \to [0,1]$, and $(f_n) \to 0$ pointwise. Prove $\displaystyle\lim_{n\to\infty} \int_0^1 f_n = 0$.
  - ▷ Simple to state using Riemann integrals. Not so easy to prove. (Challenge!)
  - ▷ Will prove this using Lebesgue integration.
    - – Riemann integration: partition the domain (count sequentially)
    - – Lebesgue integration: partition the range (stack and sort).
- **Goal:**
  - ▷ Develop Lebesgue integration.
  - ▷ Need a notion of "measure" (generalization of volume)
  - ▷ Need "σ-algebras".
- Why σ-algebras?

  Theorem 2.1 (Banach Tarski)**.** *There exists $n \in \mathbb{N}$, sets $A_1, \dots, A_n \subseteq B(0,1) \subseteq \mathbb{R}^3$ such that:*
  (1) $A_1, \dots, A_n$ partition $B(0,1)$.
  (2) There exist isometries R_i such that $R_1(A_1), \dots, R_n(A_n)$ partition $B(0,2)$.

- How do you explain this?

Definition 2.2 (σ-algebra)**.** Let X be a set. We say $\Sigma \subseteq \mathcal{P}(X)$ is a σ-algebra on X if:

(1) Nonempty: $\emptyset \in \Sigma$
(2) Closed under compliments: $A \in \Sigma \implies A^c \in \Sigma$.
(3) Closed under countable unions: $A_i \in \Sigma \implies \bigcup_{i=1}^{\infty} A_i \in \Sigma$.

Remark 2.3. Any σ-algebra is also closed under countable intersections.

Question 2.4. *Is $\mathcal{P}(X)$ is a σ-algebra?*

Question 2.5. *Is $\Sigma \overset{\text{def}}{=} \{\emptyset, X\}$ is a σ-algebra?*

Question 2.6. *Is $\Sigma = \{A \mid |A| < \infty$ or $|A^c| < \infty\}$ a σ-algebra?*

Question 2.7. *Is $\Sigma = \{A \mid$ either A or A^c is finite or countable$\}$ a σ-algebra?*

Proposition 2.8. *If $\forall \alpha \in \mathcal{A}$, Σ_α is a σ-algebra, then so is $\bigcap_{\alpha \in \mathcal{A}} \Sigma_\alpha$.*

Definition 2.9. If $\mathcal{E} \subseteq \mathcal{P}(X)$, define $\sigma(\mathcal{E})$ to be the intersection of all σ-algebras containing $\mathcal{E}$.

Remark 2.10. $\sigma(\mathcal{E})$ is the smallest σ-algebra containing $\mathcal{E}$.

Definition 2.11. Suppose X is a topological space. The *Borel σ-algebra on X* is defined to be the σ-algebra generated by all open subsets of X. Notation: $\mathcal{B}(X)$.

Question 2.12. *Can you get $\mathcal{B}(X)$ by taking all countable unions / intersections of open and closed sets?*

Question 2.13. *Is $\mathcal{B}(\mathbb{R}) = \mathcal{P}(\mathbb{R})$?*

Definition 2.14. Let Σ be a σ-algebra on X. We say μ is a (positive) measure on (X, Σ) if:

(1) $\mu \colon \Sigma \to [0, \infty]$
(2) $\mu(\emptyset) = 0$
(3) (Countable additivity): $E_1, E_2, \dots \in \Sigma$ are (countably many) pairwise disjoint sets, then $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$.

Question 2.15. *Is the second assumption necessary?*

Question 2.16. *Let $\mu(A) =$ cardinality of A. Is μ a measure?*

Question 2.17. *Fix $x_0 \in X$. Let $\mu(A) = 1$ if $x_0 \in A$, and 0 otherwise. Is μ a measure?*

Theorem 2.18. *There exists a measure λ on $\mathcal{B}(\mathbb{R}^d)$ such that $\lambda(I) = \mathrm{vol}(I)$ for all cuboids I.*

- **Goal:** Define $\int_X f \, d\mu$ (the Lebesgue integral).
- **Idea:**
  - ▷ Say $s : X \to \mathbb{R}$ is such that $s = \sum_1^N a_i \mathbf{1}_{A_i}$, for some $a_i \in \mathbb{R}$, $A_i \in \Sigma$. (Called *simple functions.*)
  - ▷ Define $\int_X s \, d\mu = \sum_1^N a_i \mu(A_i)$.
  - ▷ If $f \geqslant 0$, define $\int_X f \, d\mu = \sup_{s \leqslant f} \int_X s \, d\mu$.
- Will do this after constructing the Lebesgue measure.

3. Lebesgue Measure

3.1. Lebesgue Outer Measure.

Definition 3.1. We say $I \subseteq \mathbb{R}$ is a *cell* if I is a finite interval. Define $\ell(I) = \sup I - \inf I$.

Definition 3.2. We say $I \subseteq \mathbb{R}^d$ is a *cell* if it is a product of cells. If $I = I_1 \times \cdots \times I_d$, then define $\ell(I) = \prod_{i=1}^{d} \ell(I_i)$.

Remark 3.3. $\ell(I) = \ell(\mathring{I}) = \ell(\bar{I})$.

Remark 3.4. $\emptyset = \prod_1^d (a, a)$, and so $\ell(\emptyset) = 0$.

Remark 3.5. For all $\alpha \in \mathbb{R}^d$, $\ell(I) = \ell(I + \alpha)$.

Theorem 3.6. *There exists a (unique) measure λ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $\lambda(I) = \ell(I)$ for all cells I.*

Question 3.7. *How do you extend ℓ to other sets?*

Definition 3.8 (Lebesgue outer measure)**.** Given $A \subseteq \mathbb{R}^d$, define

$$\lambda^*(A) = \inf\Big\{ \sum_1^{\infty} \ell(I_k) \;\Big|\; A \subseteq \bigcup_1^{\infty} I_k, \text{where } I_k \text{ is a cell} \Big\}.$$

Remark 3.9. Some authors use m^* instead of λ^*.

Remark 3.10. λ^* is defined on $\mathcal{P}(\mathbb{R}^d)$; but only "well behaved" on a σ-algebra.

Question 3.11. *What is $\lambda^*(\emptyset)$? What is $\lambda^*(\mathbb{R}^d)$?*

Proposition 3.12. *If $E \subseteq F$, then $\lambda^*(E) \leqslant \lambda^*(F)$.*

Proposition 3.13. *If $E_1, E_2, \ldots \subseteq \mathbb{R}^d$, then $\lambda^*(\cup_1^\infty E_i) \leqslant \sum_1^\infty \lambda^*(E_i)$.*

Proposition 3.14. *Let $A, B \subseteq \mathbb{R}^d$, and suppose $d(A, B) > 0$. Then $\lambda^*(A \cup B) = \lambda * (A) + \lambda * (B)$.*

Proof: Only need to show $\lambda^*(A \cup B) \geqslant \lambda^*(A) + \lambda^*(B)$. If $\lambda^*(A \cup B) = \infty$, we are done, so assume $\lambda^*(A \cup B) < \infty$.

Proposition 3.15. *If $I \subseteq \mathbb{R}^d$ is a cell, then $\lambda^*(I) = \ell(I)$.*

Lemma 3.16. *If $\{I_k\}$ divide I by hyperplanes, then $\sum \ell(I_k) = \ell(I)$.*

Lemma 3.17. *$\lambda^*(A) = \inf\{\sum \ell(I_i) \mid A \subseteq \cup I_k,$ and I_k are all* open *cells$\}$.*

Proof of Proposition 3.15: Suppose first I is closed (hence compact). Pick $\varepsilon > 0$.

Proposition 3.18 (Translation invariance)**.** *For all $A \subseteq \mathbb{R}^d$, $\alpha \in \mathbb{R}^d$, $\lambda^*(A) = \lambda^*(\alpha + A)$.*

3.2. **Carathéodory Extension.** Our goal is to start with an *outer measure*, and restrict it to a *measure*.

Definition 3.19. We say μ^* is an outer measure on X if:

(1) $\mu^* : \mathcal{P}(X) \to [0, \infty]$, and $\mu^*(\emptyset) = 0$.
(2) If $A \subseteq B$ then $\mu^*(A) \leqslant \mu^*(B)$.
(3) If $A_i \subseteq X$ (not necessarily disjoint), then $\mu^*(\cup_{i=1}^\infty A_i) \leqslant \sum_{i=1}^\infty \mu^*(A_i)$.

Example 3.20. Any measure is an outer measure.

Example 3.21. The Lebesgue outer measure is an outer measure.

Theorem 3.22 (Carathéodory extension)**.** *Let $\Sigma \stackrel{\text{def}}{=} \{E \subseteq X \mid \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)\ \forall A \subseteq X\}$. Then Σ is a σ-algebra, and μ^* is a measure on (X, Σ).*

Remark 3.23. Clearly $\mu^*(A) \leqslant \mu^*(A \cap E) + \mu^*(A \cap E^c)$ for all E, A.

Intuition: Suppose $\mu^* = \lambda^*$. In order to show $\mu^*(A) \geqslant \mu^*(A \cap E) + \mu^*(A \cap E^c)$, cover A by cells so that $\mu^*(A) \geqslant \sum \ell(I_k) - \varepsilon$. Split this cover into cells that intersect E and cells that intersect E^c. If E is nice, hopefully the overlap is small.

Proof of Theorem 3.22

(1) $\emptyset \in \Sigma$.
(2) $E \in \Sigma \implies E^c \in \Sigma$.
(3) $E, F \in \Sigma \implies E \cup F \in \Sigma$. (Hence $E_1, \ldots, E_n \in \Sigma \implies \cup_1^n E_i \in \Sigma$.)
(4) If $E_1, \ldots, E_n \in \Sigma$ are pairwise disjoint, $A \subseteq X$, then $\mu^*(A \cap (\cup_1^n E_i)) = \sum_1^n \mu^*(A \cap E_i)$.
(5) Σ is closed under countable *disjoint* unions, and μ^* is countably additive on Σ.
Proof: Let $E_1, E_2, \ldots, \in \Sigma$ be pairwise disjoint, and $A \subseteq X$ be arbitrary.

Remark 3.24. Note, the above shows $\mu^*(A \cap (\cup_1^\infty E_i)) = \sum_1^\infty \mu^*(A \cap E_i)$.

Definition 3.25. Define the *Lebesgue σ-algebra* by $\mathcal{L}(\mathbb{R}^d) = \{E \mid \lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c)\ \forall A \subseteq \mathbb{R}^d\}$.

Definition 3.26. Define the *Lebesgue measure* by $\lambda(E) = \lambda^*(E)$ for all $E \in \mathcal{L}(\mathbb{R}^d)$.

Remark 3.27. By Carathéodory, $\mathcal{L}(\mathbb{R}^d)$ is a σ-algebra, and λ is a measure on $\mathcal{L}$.

Question 3.28. *Is $\mathcal{L}(\mathbb{R}^d)$ non-trivial?*

Proposition 3.29. *If $I \subseteq \mathbb{R}^d$ is a cell, then $I \in \mathcal{L}(\mathbb{R}^d)$.*

Proof:

Proposition 3.30. $\mathcal{L}(\mathbb{R}^d) \supseteq \mathcal{B}(\mathbb{R}^d)$.

Remark 3.31. We will show later that $\mathcal{L}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R}^d) + \mathcal{N}$, where $\mathcal{N} = \{A \subseteq \mathbb{R}^d \mid \lambda^*(A) = 0\}$.

Here are two results that will be proved later:

Theorem 3.32. $\mathcal{L}(\mathbb{R}^d) \supsetneq \mathcal{B}(\mathbb{R}^d)$. *(In fact the cardinality of $\mathcal{L}(\mathbb{R}^d)$ is larger than that of $\mathcal{B}(\mathbb{R}^d)$.)*

Theorem 3.33. $\mathcal{L}(\mathbb{R}^d) \subsetneq \mathcal{P}(\mathbb{R}^d)$.

Theorem 3.34 (Uniqueness)**.** *If μ is any measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $\mu(I) = \lambda(I)$ for all cells, then $\mu(E) = \lambda(E)$ for all $E \in \mathcal{B}(\mathbb{R}^d)$.*

Question 3.35. *Let $\mathcal{E} \subseteq \mathcal{P}(X)$, and suppose μ, ν are two measures which agree on $\mathcal{E}$. Must they agree on $\sigma(E)$?*

4. Abstract measures

4.1. Dynkin systems.

Question 4.1. *Say μ, ν are two measures such that $\mu = \nu$ on $\Pi \subseteq \Sigma$. Must $\mu = \nu$ on $\sigma(\Pi)$?*

▷ Clearly need Π to be closed under intersections.

Question 4.2. *Let $\Lambda = \{A \in \Sigma \mid \mu(A) = \nu(A)\}$. Must Λ be a σ-algebra?*

▷ If $A, B \in \Lambda$, must $A \cup B \in \Lambda$?
▷ If $A \subseteq B$, $A, B \in \Lambda$, must $B - A \in \Lambda$?
▷ If $A_i \subseteq A_{i+1} \in \Lambda$, must $\cup_1^\infty A_i \in \Lambda$?

Definition 4.3. We say $\Lambda \subseteq \mathcal{P}(X)$ is a λ-system if:

(1) $X \in \Lambda$
(2) If $A \subseteq B$ and $A, B \in \Lambda$ then $B - A \in \Lambda$.
(3) If $A_n \in \Lambda$, $A_n \subseteq A_{n+1}$ then $\cup_1^\infty A_n \in \Lambda$.

Definition 4.4. We say $\Pi \subseteq \mathcal{P}(X)$ is a π-system if whenever $A, B \in \Pi$, we have $A \cap B \in \Pi$.

Lemma 4.5 (Dynkin system lemma)**.** *If Π is a π-system, and $\Lambda \supseteq \Pi$, then $\Lambda \supseteq \sigma(\Pi)$.*

Corollary 4.6. *If μ, ν are finite measures such that $\mu = \nu$ on Π, and Π is closed under intersections, then $\mu = \nu$ on $\sigma(\Pi)$.*

Proof of Lemma 4.5

(1) The arbitrary intersection of λ-systems is a λ-system. So it make sense to talk about $\lambda(\Pi)$.
(2) If $\Lambda \supseteq \Pi$, then $\Lambda \supseteq \lambda(\Pi)$.
(3) If Λ is *both* a π-system and a λ-system, then Λ is a σ-algebra.
(4) To finish the proof, we only need to show $\lambda(\Pi)$ is closed under intersections.
(5) Let $C \in \lambda(\Pi)$, and define $\Lambda_C = \{B \in \lambda(\Pi) \mid B \cap C \in \lambda(\Pi)\}$. Then Λ_C is a λ-system.
(6) If $B, C \in \lambda(\Pi)$, then $B \cap C \in \lambda(\Pi)$.
  - ▹ Suppose first $D \in \Pi$. Then $D \cap B \in \lambda(\Pi)$ for all $B \in \lambda(\Pi)$.
  - ▹ For all $B \in \lambda(\Pi)$, we must have $\Lambda_B \supseteq \lambda(\Pi)$.

4.2. **Regularity of measures.**

Definition 4.7. Let X be a metric space, and μ be a Borel measure on X. We say μ is regular if:

(1) For all compact sets K, we have $\mu(K) < \infty$.
(2) For all open sets U we have $\mu(U) = \sup\{\mu(K) \mid K \subseteq U \text{ is compact}\}$.
(3) For all Borel sets A we have $\mu(A) = \inf\{\mu(U) \mid U \supseteq A,\ U \text{ open}\}$.

Motivation:

- ▹ Approximation of measurable functions by continuous functions
- ▹ Differentiation of measures
- ▹ Uniqueness in the Riesz representation theorem

Question 4.8. *If μ is regular, is $\mu(A) = \sup\{\mu(K) \mid K \subseteq A,\ K \text{ compact}\}$ for all Borel sets A?*

Remark 4.9. (1) If $X = \mathbb{R}^d$, and μ is regular, then $\mu(A) = \sup\{\mu(K) \mid K \subseteq A,\ K \text{ compact}\}$.
(2) Further, for any $\varepsilon > 0$ there exists an open set $U \supseteq A$ and a closed set $C \subseteq A$ such that $\mu(U - C) < \varepsilon$.
(3) If $\mu(A) < \infty$, then can make C above compact.

Proof. Will return and prove it using the next theorem. □

Theorem 4.10. *Suppose X is a compact metric space, and μ is a finite Borel measure on X. Then μ is regular. Further, for any $\varepsilon > 0$, there exists $U \supseteq A$ open and $K \subseteq A$ closed such that $\mu(U - K) < \varepsilon$.*

Proof:

(1) Let $\Lambda = \{A \in \mathcal{B}(X) \mid \forall \varepsilon > 0,\ \exists K \subseteq A \text{ compact},\ U \supseteq A \text{ open, such that } \mu(U - K) < \varepsilon\}$.
(2) Λ contains all open sets.
(3) Λ is a λ-system. (In this case it's easy to directly show that Λ is a σ-algebra.)
(4) Dynkin's Lemma implies $\Lambda \supseteq \mathcal{B}(X)$, finishing the proof.

Corollary 4.11. *Let X be a metric space and μ a Borel measure on X. Suppose there exists a sequence of sets $B_n \subset X$ such that $\bar{B}_n \subset \mathring{B}_{n+1}$, $\bar{B}_n$ is compact, $X = \cup_1^\infty B_n$ and $\mu(B_n) < \infty$. Then μ is regular. Further:*

(1) For any $A \in \mathcal{B}(X)$, $\mu(A) = \sup\{\mu(K) \mid K \subseteq K \text{ is compact}\}$.
(2) For any $\varepsilon > 0$, there exists $U \supseteq A$ open and $C \subseteq A$ closed such that $\mu(U - C) < \varepsilon$.

Proof. On homework. □

Corollary 4.12. *Let $A \in \mathcal{L}(\mathbb{R}^d)$.*

(1) $\lambda(A) = \inf\{\lambda(U) \mid U \supseteq A,\ U \text{ open}\} = \sup\{\lambda(K) \mid K \subseteq A,\ K \text{ compact}\}$.
(2) For any $\varepsilon > 0$, there exists $C \subseteq A$ closed and $U \supseteq A$ open such that $\lambda(U - C) < \varepsilon$.

4.3. **Non-measurable sets.**

Theorem 4.13. *There exists $E \subseteq \mathbb{R}$ such that $E \notin \mathcal{L}(R)$.*

Proof:

(1) Let $C_\alpha = \{\beta \in \mathbb{R} \mid \beta - \alpha \in \mathbb{Q}\}$. (This is the coset of $\mathbb{R}/\mathbb{Q}$ containing α.)
(2) Let $E \subseteq \mathbb{R}$ be such that $|E \cap C_\alpha| = 1$ for all α.
(3) Note if $q_1, q_2 \in \mathbb{Q}$ with $q_1 \neq q_2$, then $q_1 + E \cap q_2 + E = \emptyset$.
(4) Suppose for contradiction $E \in \mathcal{L}(\mathbb{R})$.
(5) $\lambda(E) > 0$
(6) $\lambda(E) = 0$ (contradiction).

Here are two results that we won't prove (or use later) in the interest of time.

Theorem 4.14. *Let $A \subseteq \mathbb{R}^d$. Every subset of A is Lebesgue measurable if and only if $\lambda(A^*) = 0$.*

Theorem 4.15. *There exists a set $A \subseteq \mathbb{R}$ such that:*

(1) If $E \in \mathcal{L}(\mathbb{R})$ and $E \subseteq A$, then $\lambda(E) = 0$.
(2) If $E \in \mathcal{L}(\mathbb{R})$ and $E \subseteq A^c$, then $\lambda(E) = 0$.

4.4. **Completion of measures.**

Theorem 4.16. *$A \in \mathcal{L}(\mathbb{R}^d)$ if and only if there exist $F, G \in \mathcal{B}(\mathbb{R}^d)$ such that $F \subseteq A \subseteq G$ and $\lambda(G - F) = 0$.*

Corollary 4.17. *Let $\mathcal{N} = \{A \subseteq \mathbb{R}^d \mid \lambda^*(A) = 0\}$. Then $A \in \mathcal{L}(\mathbb{R}^d)$ if and only if $A = B \cup N$ for some $B \in \mathcal{B}(\mathbb{R}^d)$ and $N \in \mathcal{N}$.*

Definition 4.18. Let $\mathcal{N} = \{A \subseteq X \mid \exists E \in \Sigma,\ E \supseteq A,\ \mu(E) = 0\}$. We say (X, Σ, μ) is complete if $\mathcal{N} \subseteq \Sigma$.

Remark 4.19. If Σ is given by Carathéodory (Theorem 3.22), then Σ is μ-complete.

Definition 4.20. Let (X, Σ, μ) be a measure space. We define the completion of Σ with respect to the measure μ by

$$\Sigma_\mu \stackrel{\text{def}}{=} \{A \subseteq X \mid \exists F, G \in \Sigma \text{ such that } F \subseteq A \subseteq G \text{ and } \mu(G - F) = 0\}$$

For every $A \in \Sigma_\mu$, find F, G as above and define $\bar{\mu}(A) = \mu(F)$.

Proposition 4.21. *Σ_μ is a σ-algebra, $\bar{\mu}$ is a measure on Σ_μ, and $(X, \Sigma_\mu, \bar{\mu})$ is complete.*

Proposition 4.22. *$\Sigma_\mu = \sigma(\Sigma \cup \mathcal{N})$, and Σ_μ is the smallest μ-complete σ-algebra containing Σ.*

Proposition 4.23. *$\mathcal{L}(\mathbb{R}^d) = \sigma(\mathcal{B}(\mathbb{R}^d) \cup \mathcal{N})$, and $\mathcal{L}(\mathbb{R}^d)$ is the completion of $\mathcal{B}(\mathbb{R}^d)$ with respect to λ.*

Remark 4.24. There could exist μ-null sets that are not in Σ.

5. Measurable Functions

5.1. Measurable functions.

Definition 5.1. Let (X, Σ, μ) be a measurable space, and (Y, τ) a topological space. We say $f: X \to Y$ is measurable if $f^{-1}(\tau) \subseteq \Sigma$.

Remark 5.2. Y is typically $[-\infty, \infty]$, $\mathbb{R}^d$, or some linear space.

Remark 5.3. Any continuous function is Borel measurable, but not conversely.

Question 5.4. *Say $f: X \to Y$ is measurable. For every $B \in \mathcal{B}(Y)$, must $f^{-1}(B) \in \Sigma$?*

Theorem 5.5. *Say $f: X \to Y$ is measurable. Then, for every $B \in \mathcal{B}(Y)$, we must have $f^{-1}(B) \in \Sigma$.*

Lemma 5.6. *Let $f: X \to Y$ be arbitrary, and Σ be a σ-algebra on X. Then $\Sigma' = \{A \subseteq Y \mid f^{-1}(A) \in \Sigma\}$ is a σ-algebra (on Y).*

Proposition 5.7. *Suppose $\mathcal{B}(Y) = \sigma(\mathcal{C})$ for some $\mathcal{C} \subseteq \mathcal{P}(Y)$. Then $f: X \to Y$ is measurable if and only if $f^{-1}(C) \in \Sigma$ for all $C \in \mathcal{C}$.*

Corollary 5.8. *Let $f: X \to [-\infty, \infty]$. Then f is measurable if and only if for all $a \in \mathbb{R}$, we have $\{f < a\} \in \Sigma$.*

Lemma 5.9. *If $f: X \to \mathbb{R}^m$ is measurable, and $g: \mathbb{R}^m \to \mathbb{R}^n$ is Borel, then $g \circ f: X \to \mathbb{R}^n$ is measurable.*

Question 5.10. *Is the above true if g was Lebesgue measurable?*

Theorem 5.11. *Let $f_n: X \to \mathbb{R}$ be a sequence of measurable functions. Then $\sup f_n$, $\inf f_n$, $\limsup f_n$, $\liminf f_n$ and $\lim f_n$ (if it exists) are all measurable.*

Lemma 5.12. *Let $f, g: X \to \mathbb{R}$. The function $(f, g): X \to \mathbb{R}^2$ is measurable if and only if both f and g are measurable.*

Corollary 5.13. *If $f, g: X \to \mathbb{R}$ are measurable, then so is $f + g$, fg and f/g (when defined).*

5.2. Cantor Function.

Definition 5.14 (Cantor function)**.** Let C be the Cantor set, and $\alpha = \log 2/\log 3$ be the Hausdorff dimension of C. Let $f(x) = H_\alpha(C \cap [0, x])/H_\alpha(C)$.

(1) $f(0) = 0$, $f(1) = 1$ and f is increasing. (In fact, f is differentiable exactly on C^c, and $f' = 0$ wherever defined.)
(2) f is continuous everywhere. (In fact f is Hölder continuous with exponent $\alpha = \log 2/\log 3$.)
(3) Let $g = f^{-1}$. That is, $g(x) = \inf\{y \mid f(y) = x\}$ (Note, since f is continuous $f(g(x)) = x$)).

Proposition 5.15. *The function $g: [0, 1] \to C$ is a strictly injective Borel measurable function.*

Theorem 5.16. $\mathcal{L}(\mathbb{R}) \supsetneq \mathcal{B}(\mathbb{R})$.

Theorem 5.17. *There exists $h_1, h_2: \mathbb{R} \to \mathbb{R}$ such that h_1 is $\mathcal{L}(\mathbb{R})$-measurable, h_2 is $\mathcal{B}(\mathbb{R})$ measurable, but $h_1 \circ h_2$ is not $\mathcal{L}(\mathbb{R})$ measurable.*

Remark 5.18. Note $h_2 \circ h_1$ has to be $\mathcal{B}(\mathbb{R})$-measurable.

5.3. Almost Everywhere.

Definition 5.19. Let (X, Σ, μ) be a measure space. We say a property P holds almost everywhere if there exists a null set N such that P holds on N^c.

Example 5.20. If f, g are two functions, we say $f = g$ almost everywhere if $\{f \neq g\}$ is a null set.

Example 5.21. Almost every real number is irrational.

Example 5.22. If $A \in \mathcal{L}(\mathbb{R})$, then $\lim_{h \to 0} \dfrac{\lambda(A \cap (x, x+h))}{h} = \mathbf{1}_A(x)$ for almost every x. (Contrast with HW3, Q3b)

Example 5.23. Let $x \in (0, 1)$, and p_n/q_n be the n^{th} convergent in the continued fraction expansion of x. Then $\lim_{n \to \infty} \frac{\log q_n}{n} = \frac{\pi^2}{12 \log 2}$.

Assume hereafter (X, Σ, μ) is complete.

Proposition 5.24. *If $f = g$ almost everywhere and f is measurable, then so is g.*

Proposition 5.25. *If $(f_n) \to f$ almost everywhere, and each f_n is measurable, then so is f.*

5.4. Approximation.

Definition 5.26. A function $s: X \to \mathbb{R}$ is called *simple* if s is measurable, and has finite range (i.e. $s(\mathbb{R}) = \{a_1, \dots a_n\}$).

Question 5.27. *Why bother with simple functions?*

Theorem 5.28. *If $f \geqslant 0$ is a measurable function, then there exists a sequence of simple functions (s_n) which increases to f.*

Corollary 5.29. *If $f: X \to \mathbb{R}$ is measurable, then there exists a sequence of simple functions (s_n) such that $(s_n) \to f$ pointwise, and $|s_n| \leqslant |f|$.*

Theorem 5.30 (Lusin)**.** *Let μ be a finite regular measure on a metric space X. Let $f: X \to \mathbb{R}$ be measurable. For any $\varepsilon > 0$ there exists a continuous function $g: X \to \mathbb{R}$ such that $\mu\{f \neq g\} < \varepsilon$.*

Lemma 5.31 (Tietze's extension theorem)**.** *If $C \subseteq X$ is closed, and $f: C \to \mathbb{R}$ is continuous, then there exist $\bar{f}: X \to \mathbb{R}$ such that $\bar{f} = f$ on C.*

Lemma 5.32. *Let $f: X \to \mathbb{R}$ be measurable. For every $\varepsilon > 0$, there exists $C \subseteq X$ closed such that $\mu(X - C) < \varepsilon$ and $f: C \to \mathbb{R}$ is continuous.*

Proof of Lusin's theorem. Previous two lemmas. □

Proof of Lemma 5.32. □

Remark 5.33. It is not true that for every measurable function f there exists a continuous function g such that $f = g$ almost everywhere.

6. Integration

6.1. **Construction of the Lebesgue integral.** Recall, $s\colon X \to \mathbb{R}$ is *simple* if s is measurable and has finite range.

Definition 6.1. Let $s \geqslant 0$ be a simple function. Let $\{a_1, \dots, a_n\} = s(X)$, and set $A_i = s^{-1}(a_i)$. Define $\int_X s\,d\mu = \sum_{i=1}^n a_i A_i$.

Remark 6.2. Always use the convention $0 \cdot \infty = 0$.

Remark 6.3. Other notation: $\int_X s\,d\mu = \int_X s(x)\,d\mu(x)$.

Proposition 6.4. *If $0 \leqslant s \leqslant t$ are simple, then $\int_X s\,d\mu \leqslant \int_X t\,d\mu$.*

Proposition 6.5. *If $s, t \geqslant 0$ are simple, then $\int_X (s+t)\,d\mu = \int_X s\,d\mu + \int_X t\,d\mu$.*

Definition 6.6. Let $f\colon X \to [0,\infty]$ be measurable. Define $\int_X f\,d\mu = \sup\{\int_X s\,d\mu \mid 0 \leqslant s \leqslant f,\ s \text{ simple.}\}$.

Definition 6.7. Let $f\colon X \to [-\infty,\infty]$ be measurable. We say f is *integrable* if $\int_X f^+\,d\mu < \infty$ and $\int_X f^-\,d\mu < \infty$. In this case we define $\int_X f\,d\mu = \int_X f^+\,d\mu - \int_X f^-\,d\mu$.

Definition 6.8. We let $L^1(X) = L^1(X,\Sigma,\mu)$ be the set of all integrable functions on X. (Note $f \in L^1 \iff |f| \in L^1$.)

Definition 6.9. We say f is *integrable in the extended sense* if *either* $\int_X f^+\,d\mu < \infty$ or $\int_X f^-\,d\mu < \infty$. In this case we still define $\int_X f\,d\mu = \int_X f^+\,d\mu - \int_X f^-\,d\mu$.

Remark 6.10. If both $\int_X f^+\,d\mu = \infty$ and $\int_X f^-\,d\mu = \infty$, then $\int_X f\,d\mu$ is not defined.

Question 6.11. *Do we have linearity?*

Proposition 6.12 (Consistency)**.** *If $s = \sum_1^n a_i \mathbf{1}_{A_i} \geqslant 0$ is simple, then $\sum a_i\mu(A_i) = \sup\{\int_X t\,d\mu \mid 0 \leqslant t \leqslant s,\ \text{simple}\}$.*

Theorem 6.13 (Monotone convergence)**.** *Say $(f_n) \to f$ almost everywhere, $0 \leqslant f_n \leqslant f_{n+1}$, then $(\int_X f_n\,d\mu) \to \int_X f\,d\mu$.*

Theorem 6.14. *If f, g are integrable, then $\int_X (f+g)\,d\mu = \int_X f\,d\mu + \int_X g\,d\mu$.*

6.2. **Dominated convergence.** When does $\lim \int_X f_n\,d\mu \neq \int_X f\,d\mu$? Two typical situations where it fails:

(1) Mass escapes to infinity
(2) Mass clusters at a point

Theorem 6.15 (Dominated convergence)**.** *Say (f_n) is a sequence of measurable functions, such that $(f_n) \to f$ almost everywhere. Moreover, there exists $F \in L^1(X)$ such that $|f_n| \leqslant F$ almost everywhere. Then $\lim_{n\to\infty} \int_X f_n\,d\mu = \int_X f\,d\mu$.*

Lemma 6.16 (Fatou)**.** *Suppose $f_n \geqslant 0$, and $(f_n) \to f$. Then $\liminf \int_X f_n\,d\mu \geqslant \int_X f\,d\mu$.*

Proof of Theorem 6.15

Theorem 6.17 (Beppo-Levi)**.** *If $f_n \geqslant 0$, then $\sum_1^\infty \int_X f_n\,d\mu = \int_X (\sum_1^\infty f_n)\,d\mu$.*

Theorem 6.18. *If $f\colon \mathbb{R}^d \to \mathbb{R}$ is Riemann integrable, then the Riemann integral of f is the same as the Lebesgue integral.*

Proof. IOU □

Question 6.19. *Let $f\colon [0,\infty) \to \mathbb{R}$ be measurable, and define the* Laplace transform *of f by $F(s) = \int_0^\infty e^{-st} f(t)\,dt$. Is F continuous? Is F differentiable?*

Question 6.20. *Let φ be a bump function, and (q_n) be an enumeration of the rationals. Define $f(x) = \sum_{n=1}^\infty \varphi(2^n(x - q_n))$. Is f finite almost everywhere?*

6.3. **Push forward measures.**

Definition 6.21. Say $f\colon X \to \mathbb{R}^d$ is integrable, then define

$$\int_X f\,d\mu = \Big(\int_X f_1\,d\mu, \dots, \int_X f_d\,d\mu\Big), \text{ where } f = (f_1, \dots, f_d).$$

Theorem 6.22. *Let (X,Σ,μ) be a measure space, $f\colon X \to Y$ be arbitrary. Define $\tau = \{A \subseteq Y \mid f^{-1}(A) \in \Sigma\}$, and define $\nu(A) = \mu(f^{-1}(A))$. Then ν is a measure on (Y,τ) and $\int_Y g\,d\nu = \int_X g \circ f\,d\mu$.*

Remark 6.23. The measure ν is called the *push forward* of μ and denoted by $f^*(\mu)$, or $\mu_{f^{-1}}$. This is used often to define Laws of random variables. (We will use it to prove the change of variable formula.)

Corollary 6.24. *If $\alpha \in \mathbb{R}^d$, then $\int_{\mathbb{R}^d} f(x+\alpha)\,d\lambda(x) = \int_{\mathbb{R}^d} f(x)\,d\lambda(x)$.*

7. Convergence

7.1. **Modes of convergence.**

Definition 7.1. We say $(f_n) \to f$ almost everywhere if for almost every $x \in X$, we have $(f_n(x)) \to f(x)$.

Definition 7.2. We say $(f_n) \to f$ in measure (notation $(f_n) \xrightarrow{\mu} f$) if for all $\varepsilon > 0$, we have $(\mu\{|f_n - f| > \varepsilon\}) \to 0$.

Definition 7.3. Let $p \in [1,\infty)$. We say $(f_n) \to f$ in L^p if $(\int_X |f_n - f|^p\,d\mu) \to 0$.

Question 7.4. *Why $p > 1$? How about $p = \infty$?*

(1) $(f_n) \to f$ almost everywhere implies $(f_n) \to f$ in measure if $\mu(X) < \infty$.
(2) $(f_n) \to f$ in measure implies $(f_n) \to f$ almost everywhere along a subsequence.
(3) $(f_n) \to f$ in L^p implies $(f_n) \to f$ in measure (for $p < \infty$), and hence $(f_n) \to f$ along a subsequence.
(4) Convergence almost everywhere or in measure don't imply convergence in L^p.

Theorem 7.5. *If $(f_n) \to f$ almost everywhere and $\mu(X) < \infty$, then $(f_n) \to f$ in measure.*

Lemma 7.6 (Egorov)**.** *If $(f_n) \to f$ almost everywhere and $\mu(X) < \infty$, for every $\varepsilon > 0$ there exists A_ε such that $\mu(A_\varepsilon^c) < \varepsilon$ and $(f_n) \to f$ uniformly on A_ε.*

Remark 7.7. This does not imply $(f_n) \to f$ uniformly almost everywhere.

Proof of Theorem 7.5

Proposition 7.8. *If $(f_n) \to f$ in measure then (f_n) need not converge to f almost everywhere.*

Proposition 7.9. *If $(f_n) \to f$ in measure, then there exists a subsequence (f_{n_k}) such that $(f_{n_k}) \to f$ almost everywhere.*

Lemma 7.10 (Borel Cantelli)**.** *If $\sum \mu(A_k) < \infty$, then almost every x belongs to only finitely many A_k (i.e. $\mu\{x \mid x \in A_k$ infinitely often$\} = 0$).*

Proof of Proposition 7.9. □

7.2. L^p spaces.

Definition 7.11. A *Banach space* is a normed vector space that is complete under the metric induced by the norm.

Example 7.12. $\mathbb{C}$, $\mathbb{R}^d$, $C(X)$, etc. are all Banach spaces.

Definition 7.13. For $p \in (0, \infty)$, define $\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{1/p}$.

Definition 7.14. For $p = \infty$, define $\|f\|_\infty = \operatorname{ess\,sup}|f| = \inf\{C \geqslant 0 \mid |f| \leqslant C$ almost surely$\}$

Definition 7.15. Let (X, Σ, μ) be a measure space, and assume Σ is μ-complete. Define $\mathcal{L}^p(X) = \{f \colon X \to \mathbb{R} \mid \|f\|_p < \infty\}$.

Question 7.16. *Is $\mathcal{L}^p(X)$ a Banach space?*

Definition 7.17. Define an equivalence relation on $\mathcal{L}^p$ by $f \sim g$ if $f = g$ almost everywhere.

Definition 7.18. Define $L^p(X) = \mathcal{L}^p(X)/\sim$.

Remark 7.19. We will always treat elements of $L^p(X)$ as functions, implicitly identifying a function with its equivalence class under the relation $\sim$. In order to be logically correct, however, we need to ensure that every operation we perform on functions respects the equivalence relation $\sim$.

Theorem 7.20. *For $p \in [1, \infty]$, $L^p(X)$ is a Banach space.*

Theorem 7.21 (Hölder's inequality)**.** *Say $p, q \in [1, \infty]$ with $1/p + 1/q = 1$. If $f \in L^p$ and $g \in L^q$, then $fg \in L^1$ and $|\int_X fg \, d\mu| \leqslant \|f\|_p \|g\|_q$.*

Remark 7.22. The relation between p and q can be motivated by dimension counting, or scaling.

Brute force proof of Theorem 7.21
Proof of Theorem 7.21 using Young's inequality.

Theorem 7.23 (Young's inequality)**.** *If $x, y \geqslant 0$, $1/p + 1/q = 1$ then $xy \leqslant x^p/p + y^q/q$.*

Lemma 7.24 (Duality)**.** *If $p \in [1, \infty)$, $1/p + 1/q = 1$, then*

$$\|f\|_p = \sup_{g \in L^q - 0} \frac{1}{\|g\|_q} \int_X fg \, d\mu = \sup_{\|g\|_q = 1} \int_X fg \, d\mu \, .$$

Remark 7.25. For $p = \infty$ this is still true if X is σ-finite.

Theorem 7.26 (Minkowski's inequality)**.** *If $f, g \in L^p$, then $f + g \in L^p$ and $\|f + g\|_p \leqslant \|f\|_p + \|g\|_p$.*

Theorem 7.27 (Jensen's inequality)**.** *If $\mu(X) = 1$, $f \in L^1(X)$, $a < f < b$ almost everywhere, and $\varphi : (a, b) \to \mathbb{R}$ is convex, then*

$$\varphi\Big(\int_X f \, d\mu\Big) \leqslant \int_X \varphi \circ f \, d\mu \, .$$

Proof of Theorem 7.20: Only remains to show L^p is complete.

Lemma 7.28. *Suppose $p < \infty$, $f_n \in L^p$ and $\sum \|f_n\|_p < \infty$. Let $f = \sum f_n$. Then $f \in L^p$, $\|f\|_p \leqslant \sum \|f_n\|_p$, $\sum f_n \to f$ in L^p and $\sum f_n \to f$ almost everywhere.*

Proof of Theorem 7.20:

Proposition 7.29. *If $p \in [1, \infty)$, $(f_n) \to f$ in L^p, then $(f_n) \to f$ in measure.*

Lemma 7.30 (Chebychev's inequality)**.** *For any $\lambda > 0$, we have $\mu(\{|f| > \lambda\}) \leqslant \dfrac{1}{\lambda} \|f\|_1$*

Proof of Proposition 7.29

7.3. Uniform integrability.

Question 7.31. *When does convergence in measure imply L^1 convergence?*

Theorem 7.32 (Vitali)**.** *Let $(f_n) \in L^1(X)$. The sequence (f_n) is convergent in L^1 if and only if*

1. *(f_n) converges in measure,*
2. *(f_n) is* uniformly integrable*,*
3. *(f_n) is tight*

Definition 7.33. Let $\{f_\alpha \mid \alpha \in \mathcal{A}\}$ be a family measurable functions on (X, Σ).

1. The family $\{f_\alpha \mid \alpha \in \mathcal{A}\}$ is *uniformly integrable* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $\mu(E) < \delta$ we have $\int_E |f| \, d\mu < \varepsilon$.
2. The family $\{f_\alpha \mid \alpha \in \mathcal{A}\}$ is *tight* if for every $\varepsilon > 0$ there exists $F \in \Sigma$ with $\mu(F) < \infty$ such that $\int_{F^c} |f_\alpha| \, d\mu < \varepsilon$ for all $\alpha \in \mathcal{A}$.

Proposition 7.34. *If $|f_\alpha| \leqslant F$ for all $\alpha \in \mathcal{A}$, and $F \in L^1$, then $\{f_\alpha \mid \alpha \in A\}$ is both uniformly integrable, and tight.*

Proof:
Proof of Theorem 7.32

Theorem 7.35. *If* $\lim_{\lambda\to\infty} \sup_n \int_{\{|f_n|>\lambda\}} |f_n|\, d\mu = 0$, *then* (f_n) *is uniformly integrable.*

Theorem 7.36. *If there exists an increasing function* $\varphi\colon [0,\infty) \to [0,\infty)$ *such that* $\lim_{x\to\infty} \frac{\varphi(x)}{x} = \infty$, *and* $\sup_n \int_X \varphi(|f_n|)\, d\mu < \infty$, *then* (f_n) *is uniformly integrable.*

Remark 7.37. The hypothesis in both the above theorems are equivalent.

Remark 7.38. If additionally $\sup_n \int_X |f_n|\, d\mu < \infty$, then the converse of both the above theorems are true.

Proof:

Corollary 7.39. *If* $(f_n) \to f$ *in measure,* $\mu(X) < \infty$, *and* $\sup_n \|f\|_p < \infty$ *for any* $p > 1$, *then* $(f_n) \to f$ *in* L^q *for every* $q \in [1, p)$.

8. Signed Measures

8.1. Hanh and Jordan Decomposition Theorems.

Definition 8.1. We say $\mu\colon \Sigma \to [-\infty, \infty]$ is a *signed measure* if:

(1) The range of μ doesn't contain *both* $+\infty$ and $-\infty$.
(2) $\mu(\emptyset) = 0$
(3) If $A_i \in \Sigma$ are countably many pairwise disjoint sets then $\mu(\cup_1^\infty A_i) = \sum_1^\infty \mu(A_i)$.

Example 8.2. Let $f \in L^1(X, \mu)$, and define ν by $\nu(A) = \int_A f\, d\mu$. Then ν is a signed measure, and we write $d\nu = f\, d\mu$.

Example 8.3. If μ, ν are two (positive) measures such that either one is finite, then $\mu - \nu$ is a signed measure.

Definition 8.4. We say $A \in \Sigma$ is a *negative set* if $\mu(B) \leqslant 0$ for all $B \in \Sigma$ with $B \subseteq A$. We say $A \in \Sigma$ is a *positive set* if $\mu(B) \geqslant 0$ for all $B \in \Sigma$ with $B \subseteq A$.

Theorem 8.5 (Hanh decomposition)**.** *If* μ *is a signed measure on* X, *then* $X = P \cup N$ *where* P *is positive and* N *is negative.*

Remark 8.6. The decomposition is unique up to null sets.

Lemma 8.7. *If* $\mu(A) \in (-\infty, \infty)$ *then there exists* $B \subseteq A$ *such that* B *is negative and* $\mu(B) \leqslant \mu(A)$.

Proof: Proof of Theorem 8.5:

Theorem 8.8 (Jordan Decomposition)**.** *Any signed measure can be expressed (uniquely) as the difference of two mutually singular positive measures.*

Definition 8.9. We say two positive measures μ, ν are *mutually singular* if there exists $C \subseteq X$ such that for every $A \in \Sigma$ we have $\mu(A \cap C) = \nu(A \cap C^c) = 0$.

Proof of Theorem 8.8

Definition 8.10. Let μ be a signed measure with Jordan decomposition $\mu = \mu^+ - \mu^-$. Define the variation of μ to be the (positive) measure $|\mu| \stackrel{\text{def}}{=} \mu^+ + \mu^-$.

Definition 8.11. Define the *total variation* of μ by $\|\mu\| = |\mu|(X)$.

Proposition 8.12. *Let* $\mathcal{M}$ *be the set of all finite signed measures on* X. *Then* $\mathcal{M}$ *is a Banach space under the total variation norm.*

8.2. Absolute Continuity.

Definition 8.13. Let μ, ν be two measures. We say ν is absolutely continuous with respect to μ (notation $\nu \ll \mu$) if whenever $\mu(A) = 0$ we have $\nu(A) = 0$.

Example 8.14. Let $g \geqslant 0$ and define $\nu(A) = \int_A g\, d\mu$. (Notation: Say $d\nu = g\, d\mu$.)

Theorem 8.15 (Radon-Nikodym)**.** *If* μ, ν *are two positive* σ*-finite measures such* $\nu \ll \mu$, *then there exists a unique measurable function* g *such that* $0 \leqslant g < \infty$ *almost everywhere and* $d\nu = g\, d\mu$.

Theorem 8.16 (Lebesgue Decomposition)**.** *Let* μ, ν *be positive measures such that* ν *is* σ*-finite. There exists a unique pair of measures* (ν_{ac}, ν_s) *such that* $\nu_{ac} \ll \mu$, $\nu_s \perp \mu$, *and* $\nu = \nu_{ac} + \nu_s$.

Corollary 8.17. *Let* μ *be a positive measure, and* ν *be a finite signed measure. There exists a unique pair of signed measures* (ν_{ac}, ν_s) *such that* $\nu_{ac} \ll \mu$, $\nu_s \perp \mu$ *and* $\nu = \nu_{ac} + \nu_s$.

Corollary 8.18. *Let* μ, ν *be* σ*-finite positive measures. There exists a unique positive measure* ν_s *and nonnegative measurable function* g *such that* $\mu \perp \nu_s$ *and* $d\nu = d\nu_s + g\, d\mu$.

8.3. Dual of L^p.

Proposition 8.19. *Let* U, V *be Banach spaces, and* $T\colon U \to V$ *be linear. Then* T *is continuous if and only if there exists* $c < \infty$ *such that* $\|Tu\|_V \leqslant c\|u\|_U$ *for all* $u \in U$, $v \in V$.

Definition 8.20. We say $T\colon U \to V$ is a bounded linear transformation if T is linear and there exists $c < \infty$ such that $\|Tu\|_V \leqslant c\|u\|_U$ for all $u \in U$, $v \in V$.

Definition 8.21. The *dual* of U is defined by

$$U^* = \{u^* \mid u^*\colon U \to \mathbb{R} \text{ is bounded and linear.}\}$$

Define a norm on U^* by

$$\|u^*\|_{U^*} \stackrel{\text{def}}{=} \sup_{u \in U - 0} \frac{1}{\|u\|_U} u^*(u) = \sup_{\|u\|_U = 1} \frac{1}{\|u\|_U} u^*(u) = \sup_{\|u\|_U = 1} \frac{1}{\|u\|_U} |u^*(u)|\,.$$

Proposition 8.22. *The dual of a Banach space is a Banach space.*

Proposition 8.23. *Let* $1/p + 1/q = 1$, $g \in L^q(X)$. *Define* $T_g\colon L^p \to \mathbb{R}$ *by* $T_g f = \int_X f g\, d\mu$. *Then* $T_g \in (L^p)^*$.

Proposition 8.24. *The map* $g \mapsto T_g$ *is a bounded linear map from* $L^q \to (L^p)^*$.

Theorem 8.25. *Let* (X, Σ, μ) *be a* σ*-finite measure space,* $p \in [1, \infty)$, $1/p + 1/q = 1$. *The map* $g \mapsto T_g$ *is a bijective linear isometry between* L^q *and* $(L^p)^*$.

Remark 8.26. For $p \in (1, \infty)$ the above is still true even if X is not σ-finite.

Remark 8.27. For $p = \infty$, the map $g \mapsto T_g$ gives an *injective* linear isometry of $L^1 \to (L^\infty)^*$). It is not surjective in most cases.

8.4. Riesz Representation Theorem.

Theorem 8.28 (Riesz Representation Theorem)**.** *Let X be a compact metric space, and $\mathcal{M}$ be the set of all finite signed measures on X. Define $\Lambda\colon \mathcal{M} \to C(X)^*$ by $\Lambda_\mu(f) = \int_X f\,d\mu$ for all $\mu \in \mathcal{M}$ and $f \in C(X)$. Then Λ is a bijective linear isometry.*

Remark 8.29. In particular, for every $I \in C(X)^*$, there exists a unique finite regular Borel measure μ such that $I(f) = \int_X f\,d\mu$ for every $f \in C(X)$.

9. Product measures

9.1. Fubini and Tonelli theorems. Let (X, Σ, μ) and (Y, τ, ν) be two measure spaces. Define $\Sigma \times \tau = \{A \times B \mid A \in \Sigma,\ B \in \tau\}$, and $\Sigma \otimes \tau = \sigma(\Sigma \times \tau)$.

Theorem 9.1. *Let μ, ν be two σ-finite measures. There exists a unique measure π on $\Sigma \otimes \tau$ such that $\pi(A \times B) = \mu(A)\nu(B)$ for every $A \in \Sigma$, $B \in \tau$.*

Theorem 9.2 (Tonelli)**.** *Let $f\colon X \times Y \to [0, \infty]$ be $\Sigma \otimes \tau$-measurable. For every $x_0 \in X$, $y_0 \in Y$ the functions $x \mapsto f(x, y_0)$ and $y \mapsto f(x_0, y)$ are measurable. Moreover,*

$$\int_{X\times Y} f(x,y)\,d\pi(x,y) = \int_{x\in X}\Big(\int_{y\in Y} f(x,y)\,d\nu(y)\Big)\,d\mu(x)$$

$$= \int_{y\in Y}\Big(\int_{x\in X} f(x,y)\,d\mu(x)\Big)\,d\nu(y)\,. \tag{9.1}$$

Theorem 9.3 (Fubini)**.** *If $f \in L^1(X \times Y, \pi)$ then for almost every $x_0 \in X$, $y_0 \in Y$, the functions $x \mapsto f(x, y_0)$ and $y \mapsto f(x_0, y)$ are integrable in x and y respectively. Moreover,(9.1) holds.*

Lemma 9.4. *For every $E \subseteq X \times Y$, $x \in X$, $y \in Y$ define the horizontal and vertical slices of E by $H_y(E) = \{x \in X \mid (x, y) \in E\}$ and $V_x(E) = \{y \in Y \mid (x, y) \in E\}$.*

(1) For every $x \in X$, $y \in Y$ we have $H_y(E) \in \Sigma$ and $V_x(E) \in \tau$.
(2) The functions $x \mapsto \nu(V_x(E))$ and $y \mapsto \mu(H_y(E))$ are measurable.

Proof of Theorem 9.1
Proof of Theorem 9.2
Proof of Theorem 9.3

Theorem 9.5 (Layer Cake)**.** *If $f\colon X \to [0, \infty]$ is measurable then $\displaystyle\int_X f\,d\mu = \int_0^\infty \mu(f > t)\,dt$.*

Proposition 9.6. *If $(a_{m,n})$ are such that $\displaystyle\sum_{m,n=0}^{\infty} |a_{m,n}| < \infty$, then $\displaystyle\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{m,n} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{m,n}$.*

Theorem 9.7 (Minkowski's inequality)**.** *If $f\colon X \times Y \to \mathbb{R}$ is measurable, then*

$$\Big(\int_X \Big|\int_Y f(x,y)\,d\nu(y)\Big|^p\,d\mu(x)\Big)^{1/p} \leqslant \int_Y \Big(\int_X |f(x,y)|^p\,d\mu(x)\Big)^{1/p}\,d\nu(y)$$

9.2. Convolutions.

Definition 9.8. If $f, g \in L^1(\mathbb{R}^d)$ define the *convolution* by $\displaystyle f * g(x) = \int_{\mathbb{R}^d} f(x - y)g(y)\,dy = \int_{\mathbb{R}^d} f(y)g(x-y)\,dy$.

Remark 9.9. If $f, g \in L^1(\mathbb{R}^d)$, then $f * g < \infty$ almost everywhere.

Theorem 9.10 (Young)**.** *If $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ then $f * g \in L^r(\mathbb{R}^d)$, and $\|f * g\|_{L^r} \leqslant \|f\|_{L^p}\|g\|_{L^q}$.*

Remark 9.11. One can show $\|f * g\|_r \leqslant C_{p,q}\|f\|_p\|g\|_q$ for some constant $C_{p,q} < 1$. The optimal constant can be found by choosing f, g to be Gaussian's.

Definition 9.12. (φ_n) is an *approximate identity* if: (1) $\varphi_n \geqslant 0$, (2) $\int_{\mathbb{R}^d} \varphi_n = 1$, and (3) $\forall \varepsilon > 0$, $\displaystyle\lim_{n\to\infty}\int_{\{|y|>\varepsilon\}} \varphi_n(y)\,dy = 0$.

Example 9.13. Let $\varphi \geqslant 0$ be any function with $\int_{\mathbb{R}^d} \varphi = 1$, and set $\varphi_\varepsilon = \frac{1}{\varepsilon^d}\varphi(\frac{x}{\varepsilon})$.

Example 9.14. $G_t(x) = (2\pi t)^{-d/2}\exp(-|x|^2/(2t))$, for $x \in \mathbb{R}^d$.

Proposition 9.15. *If $p \in [1, \infty)$, $f \in L^p$, and (φ_n) is an approximate identity, then $\varphi_n * f \to f$ in L^p.*

Remark 9.16. For $p = \infty$ the above is still true at points where f is continuous.

9.3. Fourier Series. Let $X = [0, 1]$ with the Lebesgue measure. For $n \in \mathbb{Z}$ define $e_n(x) = e^{2\pi i n x}$, and given $f, g \in L^2(X, \mathbb{C})$ define $\langle f, g\rangle = \int_X f\bar{g}\,d\lambda$. This defines an *inner product* on $L^2(X)$, and $\|f\|_{L^2}^2 = \langle f, f\rangle$.

Definition 9.17. If $f \in L^2$, $n \in \mathbb{Z}$, define the n^{th} Fourier coefficient of f by $\hat{f}(n) = \langle f, e_n\rangle$.

Definition 9.18. For $N \in \mathbb{N}$, let $S_N f = \sum_{-N}^{N} \hat{f}(n)e_n$, be the N-th partial sum of the *Fourier Series* of f.

Question 9.19. *Does $S_N f \to f$? In what sense?*

Lemma 9.20. $\langle e_n, e_m\rangle = \delta_{n,m}$.

Corollary 9.21. *Let $p \in \operatorname{span}\{e_{-N}, \dots, e_N\}$. Then $\langle f - S_N f, p\rangle = 0$. Consequently, $\|f - S_N f\|_2 \leqslant \|f - p\|_2$.*

Proposition 9.22. *$S_N f = D_N * f$, where $\displaystyle D_N = \frac{\sin(2\pi(N + \frac{1}{2})x)}{\sin(\pi x)}$. The functions D_N are called the* Dirichlet Kernels.

Proposition 9.23. *Define the* Cesàro sum *by $\sigma_N f = \frac{1}{N}\sum_0^{N-1} S_n f$. Then $\sigma_N f = F_N * f$, where $\displaystyle F_N = \frac{1}{N}\Big(\frac{\sin(N\pi x)}{\sin(\pi x)}\Big)^2$.*

Remark 9.24. The functions F_N are called the *Fejér Kernels.*

Proposition 9.25. *The Fejér kernels are an approximate identity, but the Dirichlet kernels are not.*

Corollary 9.26. *If $p \in [1,\infty)$ and $f \in L^p$, then $\sigma_N f \to f$ in L^p.*

Corollary 9.27. *If $f \in L^2$ then $S_N f \to f$ in L^2.*

Theorem 9.28. *If $p \in (1,\infty)$, $f \in L^p$ then $S_N f \to f$ in L^p.*

Proof. The proof requires boundedness of the Hilbert transform and is beyond the scope of this course. □

Theorem 9.29. *If $f \in L^\infty$ and is Hölder continuous at x with any exponent $\alpha > 0$, then $S_n f(x) \to x$.*

Proof. On homework. □

Remark 9.30. If f is simply continuous at x, then certainly $\sigma_n f(x) \to f(x)$, but $S_n f(x)$ need not converge to $f(x)$. In fact, for almost every continuous periodic function function, $S_N f$ diverges on a dense G_δ.

The next few results establish a connection between the regularity (differentiability) of a function and decay of its Fourier coefficients.

Theorem 9.31 (Riemann Lebesgue)**.** *Let μ be a finite measure and set $\hat\mu(n) = \int_0^1 \bar{e_n}\, d\mu$. If $\mu \ll \lambda$, then $(\hat\mu(n)) \to 0$ as $n \to \infty$.*

Theorem 9.32 (Parseval's equality)**.** *If $f \in L^2([0,1])$ then $\|\hat f\|_{\ell^2} = \|f\|_{L^2}$.*

Question 9.33. *What are the Fourier coefficients of f'?*

Definition 9.34. We say g is a *weak derivative* of f if $\langle f, \varphi' \rangle = -\langle g, \varphi \rangle$ for all $\varphi \in C^\infty_{per}([0,1])$.

Proposition 9.35. *If $f \in L^1$ has a weak derivative $f' \in L^1$, then $(f')\hat{}(n) = 2\pi i n \hat f(n)$.*

Corollary 9.36. *If $f \in L^2$ has a weak derivative $f' \in L^2$, then $\sum [(1+|n|)|\hat f(n)|]^2 < \infty$.*

Definition 9.37. For $s \geqslant 0$, let $H^s_{per} \stackrel{\text{def}}{=} \{f \in L^2 \mid \|f\|_{H^s} < \infty\}$, where $\|f\|^2_{H^s} = \sum (1+|n|)^{2s} |\hat f(n)|^2$.

Remark 9.38. H^s is essentially the space of L^2 functions that also have s "weak derivatives" in L^2.

Theorem 9.39 (1D Sobolev Embedding)**.** *If $s > \frac12$ and $H^s_{per} \subseteq C_{per}([0,1])$ and the inclusion map is continuous.*

Remark 9.40. Need $s > \frac12$. The theorem is false when $s = 1/2$.

Remark 9.41. In d dimensions the above is still true if you assume $s > d/2$.

Remark 9.42. More generally one can show for $\alpha \in (0,1)$, $s = \frac12 + n + \alpha$, $H^s_{per} \subseteq C^{n,\alpha}$.

Theorem 9.43 (1D Sobolev embedding)**.** *If $s > \frac12 - \frac{1}{2n}$, then $H^s_{per} \subseteq L^{2n}$ and the inclusion map is continuous.*

Remark 9.44. The above is true for $s = \frac12 - \frac1p$, for some $p \in [1,\infty)$ but our proof won't work.

10. Differentiation

10.1. Lebesgue Differentiation.

Theorem 10.1 (Fundamental theorem of Calculus 1)**.** *If f is continuous and $F(x) = \int_0^x f(t)\,dt$, then F is differentiable and $F' = f$.*

Theorem 10.2 (Fundamental theorem of Calculus 2)**.** *If f is Riemann integrable, and $F' = f$, then $\int_a^b f = F(b) - F(a)$.*

Our goal is to generalize these to Lebesgue integrable functions.

Theorem 10.3 (Lebesgue Differentiation)**.** *If $f \in L^1(\mathbb{R}^d)$, then for almost every $x \in \mathbb{R}^d$, $\displaystyle\lim_{\varepsilon\to 0} \frac{1}{|B(x,\varepsilon)|} \int_{B(x,\varepsilon)} f\, d\lambda = f(x)$.*

Lemma 10.4 (Vitali Covering Lemma)**.** *Let $W \subseteq \cup_1^N B(x_i, r_i)$. There exists $S \subseteq \{1,\dots,N\}$ such that:*

1. *$\{B(x_i,r_i) \mid i \in S\}$ are pairwise disjoint.*
2. *$W \subseteq \cup_{i\in S} B(x_i, 3r_i)$ and hence $|W| \leqslant 3^d \sum_{i\in S} B(x_i, r_i)$.*

Definition 10.5 (Maximal function)**.** Let μ be a finite (signed) Borel measure on $\mathbb{R}^d$. Define the *maximal function* of μ by

$$M\mu(x) = \sup_{r>0} \frac{|\mu|(B(x,r))}{|B(x,r)|}$$

Proposition 10.6. *$M\mu \in L^{1,\infty}$, and $|M\mu > \alpha| \leqslant \frac{3^d}{\alpha}\|\mu\|$.*

Corollary 10.7. *If $f \in L^1(\mathbb{R}^d)$, then $|\{Mf > \alpha\}| \leqslant \frac{3^d}{\alpha}\|f\|_{L^1}$.*

Proposition 10.8. *If $f \in L^1(\mathbb{R}^d)$, then $\displaystyle\lim_{r\to 0} \frac{1}{|B(x,r)|}\int_{|y-x|<r} |f(y) - f(x)|\,dy = 0$ almost everywhere.*

Remark 10.9. This immediately implies Theorem 10.3.

Corollary 10.10. *If $\mu \ll \lambda$ is a finite signed measure, then the Radon-Nikodym derivative is given by $\displaystyle\frac{d\mu}{d\lambda} = \lim_{r\to 0}\frac{\mu(B(x,r))}{|B(x,r)|}$.*

Remark 10.11. Will use this to prove the change of variables formula.

Theorem 10.12. *Let μ be a finite signed measure which is mutually singular with respect to the Lebesgue measure.*

1. *$\frac{d\mu}{d\lambda} = 0$, λ-almost everywhere.*
2. *$\frac{d|\mu|}{d\lambda} = \infty$, $|\mu|$-almost everywhere.*

10.2. **Fundamental theorem of calculus.**

Question 10.13. *Does $f\colon [0,1] \to \mathbb{R}$ differentiable almost everywhere imply $f' \in L^1$?*

Question 10.14. *Does $f\colon [0,1] \to \mathbb{R}$ differentiable almost everywhere, and $f' \in L^1$ imply $f(x) = \int_0^x f'$?*

Definition 10.15. We say $f\colon \mathbb{R} \to R$ is absolutely continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every finite disjoint collection of intervals (x_1, y_1), ..., (x_N, y_N), such that $\sum_1^N |x_i - y_i| < \delta$, we have $\sum_1^N |f(x_i) - f(y_i)| < \varepsilon$.

Remark 10.16. Any absolutely continuous function is continuous, but not conversely.

Theorem 10.17. *Let $f\colon [a,b] \to \mathbb{R}$ be measurable. Then f is absolutely continuous if and only if f is differentiable almost everywhere, $f' \in L^1$, and $f(x) - f(a) = \int_a^x f'$ everywhere.*

Proof of the reverse implication of Theorem 10.17

Lemma 10.18. *If f is absolutely continuous, monotone and injective, then f is differentiable almost everywhere, $f' \in L^1$ and $f(x) - f(a) = \int_a^x f'$ everywhere.*

Lemma 10.19. *If f is absolutely continuous and monotone, then f is differentiable almost everywhere, $f' \in L^1$ and $f(x) - f(a) = \int_a^x f'$ almost everywhere.*

Lemma 10.20. *If f is absolutely continuous then there exist g, h increasing such that $f = g - h$.*

Proof of the forward implication of Theorem 10.17. Follows immediately from the previous lemmas. ☐

10.3. **Change of variables.**

Theorem 10.21. *Let $U, V \subseteq \mathbb{R}^d$ be open and $\varphi\colon U \to V$ be C^1 and bijective. If $f \in L^1(V)$, then $\displaystyle\int_V f\, d\lambda = \int_U f \circ \varphi |\det \nabla\varphi|\, d\lambda$.*

The main idea behind the proof is as follows: Let $\mu(A) = \lambda(\varphi(A))$.

Lemma 10.22. *μ is a Borel measure and $\int_U f \circ \varphi\, d\mu = \int_V f\, d\lambda$.*

Lemma 10.23. *$\mu \ll \lambda$*

Lemma 10.24. *$D\mu = |\det \nabla\varphi|$, where $D\mu(x) = \displaystyle\lim_{r\to 0} \frac{\mu(B(x,r))}{|B(x,r)|}$.*

Proof of Theorem 10.21. Follows immediately from the above Lemmas. ☐

Proof of Lemma 10.22
Proof of Lemma 10.23
Proof of Lemma 10.24

11. **Fourier Transform**

11.1. **Definition and Basic Properties.**

(1) Recall if $f \in L^2_{per}([0,1])$, we set $e_n(x) = e^{2\pi i n x}$, $a_n = \int_0^1 f(x) e^{-2\pi i n x}\, dx$ and got $f = \sum a_n e_n$ in L^2.
(2) Suppose now $f \in L^2_{per}([-L/2, L/2])$. Can we rescale and send $L \to \infty$?

Definition 11.1. If $f \in L^1(\mathbb{R}^d)$, $\xi \in \mathbb{R}^d$, define the *Fourier transform* of f (denoted by $\hat f$) by $\displaystyle\hat f(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \xi\rangle}\, dx$

Remark 11.2. More generally, if μ is a finite (signed) Borel measure, then can define $\displaystyle\hat\mu(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i\langle x,\xi\rangle}\, d\mu(x)$.

Analogous to Fourier series, we will show that $\hat f$ is defined even for $f \in L^2$, and prove $\displaystyle f(x) = \int_{\mathbb{R}^d} \hat f(\xi) e^{+2\pi i \langle x,\xi\rangle}\, d\xi$.

Lemma 11.3 (Linearity)**.** *If $f, g \in L^1$, $\alpha \in \mathbb{R}$ then $(f + \alpha g)^\wedge = \hat f + \alpha \hat g$.*

Lemma 11.4 (Translation)**.** *Let $\tau_y f(x) = f(x - y)$. Then*

$$(\tau_y f)^\wedge(\xi) = e^{-2\pi i \langle y,\xi\rangle} \hat f(\xi)\,.$$

Lemma 11.5 (Dilations)**.** *Let $\delta_\lambda f(x) = \frac{1}{\lambda^d} f(\frac{x}{\lambda})$. Then $(\delta_\lambda f)^\wedge(\xi) = \hat f(\lambda\xi)$.*

Lemma 11.6. *If $f, g \in L^1$, then $(f * g)^\wedge = \hat f \hat g$.*

Lemma 11.7. *If $(1 + |x|) f(x) \in L^1(\mathbb{R}^d)$ then $\partial_j \hat f(\xi) = (-2\pi i x_j f(x))^\wedge(\xi)$.*

Lemma 11.8. *If $f \in C^1_0$, $\partial_j f \in L^1$, then $(\partial_j f)^\wedge(\xi) = 2\pi i \xi_j \hat f(\xi)$.*

Theorem 11.9 (Riemann-Lebesgue Lemma)**.** *If $f \in L^1$, then $\hat f \in C_0$ and $\|\hat f\|_{L^\infty} \leqslant \|f\|_{L^1}$.*

11.2. **Fourier Inversion.**

Theorem 11.10 (Inversion)**.** *If $f, \hat f \in L^1$, then $\displaystyle f(x) = \int_{\mathbb{R}^d} \hat f(\xi) e^{+2\pi i\langle x,\xi\rangle}\, d\xi$.*

Direct proof attempt:

Lemma 11.11. *If $G(x) = (2\pi)^{-d/2} e^{-|x|^2/2}$, then $\hat G(\xi) = e^{-|2\pi\xi|^2/2}$, and hence $\hat{\hat G} = G$.*

Lemma 11.12. *If $f, g \in L^1$ then $\int_{\mathbb{R}^d} f \hat g = \int_{\mathbb{R}^d} \hat f g$.*

Lemma 11.13. *If $f \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and $\hat f \in L^1(\mathbb{R}^d)$, then*

$$f(x) = \int_{\mathbb{R}^d} \hat f(\xi) e^{+2\pi i\langle x,\xi\rangle}\, d\xi\,.$$

Proof of Theorem 11.10.

Remark 11.14. If $f, \hat f \in L^1$, then $\|f - \varphi_\varepsilon * f\|_{L^\infty} \leqslant \|\hat f - (\varphi_\varepsilon * f)^\wedge\|_{L^1} \to 0$

Remark 11.15. If $f, \hat f \in L^1$ then $\hat{\hat f}(x) = f(-x)$.

11.3. L^2-theory.

Theorem 11.16 (Plancherel)**.** *The Fourier transform extends to a bijective linear isometry on $L^2(\mathbb{R}^d;\mathbb{C})$.*

Definition 11.17. Define the *Schwartz space*, $\mathcal{S}$, to be the set of all smooth functions such that $\sup_x(1+|x|^n)|D^\alpha f(x)| < \infty$ for all $n \in \mathbb{N}$ and multi-indexes α.

Remark 11.18. Note $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{S}$, and so $\mathcal{S}$ is a dense subset of $L^p(\mathbb{R}^d)$ for all $p \in [1,\infty)$.

Lemma 11.19. *If $f, g \in \mathcal{S}$, then $\int_{\mathbb{R}^d} f\,\bar{g}\,dx = \int_{\mathbb{R}^d} \hat{f}\,\bar{\hat{g}}\,d\xi$.*

Proof of Theorem 11.16

Definition 11.20. Let $s \geqslant 0$ and define the *Sobolev space of index s* by

$$H^s = \{f \in L^2(\mathbb{R}^d) \mid \|f\|_{H^s} < \infty\}, \quad \text{where } \|f\|_{H^s} = \Big(\int_{\mathbb{R}^d} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi\Big)^{1/2}.$$

Remark 11.21. A function $f \in H^1$ if and only if f and all first order *weak derivatives* are in L^2.

Remark 11.22. For $s < 0$, one needs to define H^s as the completion of $\mathcal{S}$ under the H^s norm.

Proposition 11.23. *Let $s \in (0,1)$. Then $f \in H^s$ if and only if*

$$\int_0^\infty \Big(\frac{\|f - \tau_h f\|_{L^2}}{|h|^s}\Big)^2 \frac{dh}{h^d} < \infty\,.$$

Remark 11.24. For $s = 1$, we instead need $\sup_{h\neq 0} \frac{1}{|h|}\|f - \tau_h f\|_{L^2} < \infty$.

Remark 11.25. If $s \in (0,1]$, then there exists $C = C(s)$ such that $\|f - \tau_h f\|_{L^2} \leqslant C|h|^s\|f\|_{L^2}$ for all $f \in H^s$, $h \in \mathbb{R}^d$.

Theorem 11.26 (Sobolev embedding)**.** *If $s > d/2$ then $H^s(\mathbb{R}^d) \subseteq C_b(\mathbb{R}^d)$, and the inclusion map is continuous.*

Corollary 11.27. *If $s > n + d/2$, then $H^s(\mathbb{R}^d) \subseteq C_b^n(\mathbb{R}^d)$ and the inclusion map is continuous.*

Proposition 11.28 (Elliptic regularity)**.** *Say $f \in \mathcal{S}(\mathbb{R}^d)$, $u \in H^2(\mathbb{R}^d)$ is such that $\lim_{|x|\to\infty} |x|^d |\nabla u(x)| = 0$ and $-\Delta u = f$, then $u \in \mathcal{S}$.*

Appendix A. The d-dimensional Hausdorff measure in $\mathbb{R}^d$

Let (X,d) be any metric space, $\delta > 0$, $\alpha \geqslant 0$ and $H^*_{\alpha,\delta}$ be the outer measure defined by

$$H^*_{\alpha,\delta}(A) = \inf\Big\{\sum_1^\infty \rho_\alpha(E_i) \,\Big|\, \operatorname{diam}(E_i) < \delta\,,\ \text{and } A \subset \bigcup_1^\infty E_j\Big\}\,,$$

where

$$\rho_\alpha(A) = \frac{\pi^{\alpha/2}}{\Gamma(1+\frac{\alpha}{2})}\Big(\frac{\operatorname{diam}(A)}{2}\Big)^\alpha.$$

Remark A.1. The function ρ_α above are chosen so that if $A = B(0,r) \subseteq \mathbb{R}^d$, then $\rho_d(A) = |A|$.

Definition A.2. Let $H^*_\alpha = \lim_{\delta\to 0} H^*_\alpha$.

Proposition A.3 (From homework 2)**.** *The outer measure H^*_α restricts to a measure on the Borel σ-algebra.*

Theorem A.4. *If $X = \mathbb{R}^d$, and $\alpha = d$ then $H_\alpha = \lambda$ (the Lebesgue measure).*

Lemma A.5 (Infinite Vitali's Covering Lemma)**.** *Let $W \subseteq \cup_{\alpha\in A} B(x_\alpha, r_\alpha)$, with $\sup r_\alpha < \infty$. There exists a countable set $\mathcal{I} \subseteq A$ such that:*

(1) $\{B(x_i, r_i) \mid i \in \mathcal{I}\}$ are pairwise disjoint.
(2) $W \subseteq \cup_{i\in\mathcal{I}} B(x_i, 5r_i)$ and hence $|W| \leqslant 5^d \sum_{i\in S} B(x_i, r_i)$.

Lemma A.6. *Let $U \subseteq \mathbb{R}^d$ be open and $\delta > 0$. There exists countably many $x_i \in U$, $r_i \in (0,\delta)$ such that $\overline{B(x_i,r_i)} \subseteq U$, are pairwise disjoint, and $|U - \cup\overline{B(x_i,r_i)}| = 0$.*

Lemma A.7. $H_d \leqslant \lambda$.

Theorem A.8 (Isodiametric inequality)**.**

$$|A| \leqslant |B(0,1/2)|\operatorname{diam}(A)^d = |B(0,\operatorname{diam}(A)/2)|$$

Remark A.9. Note A need not be contained in a ball of radius $\operatorname{diam}(A)/2$.

Proof of Theorem A.4.

Proposition A.10 (Steiner Symmetrization)**.** *Let $P \subseteq \mathbb{R}^d$ be a hyperplane with unit normal $\hat{n}$. Let $A \in \mathcal{L}(\mathbb{R}^d)$. There exists $S_P(A) \in \mathcal{L}(\mathbb{R}^d)$ such that:*

(1) $S_P(A)$ is symmetric about P (i.e. for any $x \in P$, $t \in \mathbb{R}$, we have $x + t\hat{n} \in S_P(A) \iff x - t\hat{n} \in S_P(A)$).
(2) $\operatorname{diam}(S_P(A)) \leqslant \operatorname{diam}(A)$.
(3) $|S_P(A)| = |A|$.

Proof of Theorem A.8