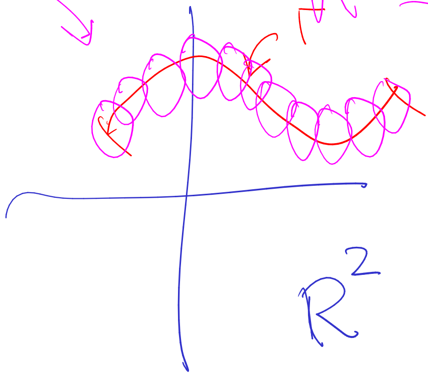
$\boxed{\text{Slides from today are in chat}}$

Last time: $\exists!$ Borel measure $\lambda \ni \lambda(I) = \ell(I)$ $\forall$ cells I.

cover with balls of (small diam)



$\Gamma \subseteq \mathbb{R}^2$ a curve.

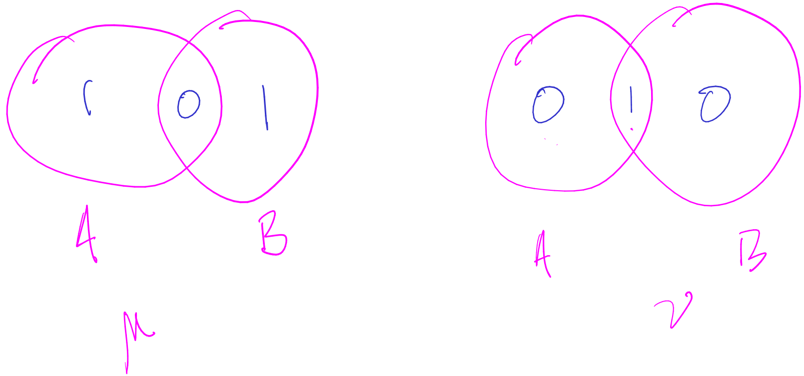
$\lambda(\Gamma) = 0$

4. **Abstract measures**

4.1. **Dynkin systems.**

Question 4.1. *Say μ, ν are two measures such that $\mu = \nu$ on $\Pi \subseteq \Sigma$. Must $\mu = \nu$ on $\sigma(\Pi)$?*

▷ Clearly need Π to be closed under intersections.



$\mu(A) = \nu(A) = 1$

$\mu(B) = \nu(B) = 1$

$\mu(A \cap B) \neq \nu(A \cap B)$

Question 4.2. *Let $\Lambda = \{A \in \Sigma \mid \mu(A) = \nu(A)\}$. Must Λ be a σ-algebra?*

- $\triangleright$ If $A, B \in \Lambda$, must $A \cup B \in \Lambda$? Stuck
- $\triangleright$ If $A \subseteq B$, $A, B \in \Lambda$, must $B - A \in \Lambda$? Yes
- $\triangleright$ If $A_i \subseteq A_{i+1} \in \Lambda$, must $\cup_1^\infty A_i \in \Lambda$? YES.

(Given $\mu(A) = \nu(A)$ $\forall A \in \Pi$

i.e. Given $\Lambda \supseteq \Pi$)

WTS $\Lambda \supseteq \sigma(\Pi)$

$\boxed{\text{need finiteness}}$

$$\mu(B-A) = \mu(B) - \mu(A) = \nu(B) - \nu(A) = \nu(B-A)$$

Yes: $\mu\left(\bigcup_1^\infty A_i\right) = \lim_{i\to\infty} \mu(A_i) = \lim \nu(A_i) = \nu\left(\bigcup_1^\infty A_i\right)$.

Cont. from

Definition 4.3. We say $\Lambda \subseteq \mathcal{P}(X)$ is a λ-system if:

(1) $X \in \Lambda$

(2) If $A \subseteq B$ and $A, B \in \Lambda$ then $B - A \in \Lambda$.

(3) If $A_n \in \Lambda$, $A_n \subseteq A_{n+1}$ then $\cup_1^\infty A_n \in \Lambda$.

($\Rightarrow$ Λ is closed under complements)

Definition 4.4. We say $\Pi \subseteq \mathcal{P}(X)$ is a π-system if whenever $A, B \in \Pi$, we have $\boxed{A \cap B \in \Pi.}$

Lemma 4.5 (Dynkin system lemma)**.** *If* Π *is a* π*-system, and* $\Lambda \supseteq \Pi$*, then* $\Lambda \supseteq \sigma(\Pi)$.

(Λ is a λ system).

Corollary 4.6. *If* μ, ν *are* $\boxed{\textit{finite}}$ *measures such that* $\mu = \nu$ *on* Π*, and* Π *is closed under intersections, then* $\mu = \nu$ *on* $\sigma(\Pi)$.

$\hookrightarrow$ Pf of cor: Above $\Rightarrow$ $\{A \mid \mu(A) = \nu(A)\}$ is a λ-sys.

Dynkin $\Rightarrow$ $\{A \mid \mu(A) = \nu(A)\} \supseteq \sigma(\Pi)$

$\Rightarrow$ $\mu = \nu$ on $\sigma(\Pi)$ QED

You check.

λ-sys gen by Π

Proof of Lemma 4.5

(1) The arbitrary intersection of λ-systems is a λ-system. So it make sense to talk about $\lambda(\Pi)$.

(2) If $\Lambda \supseteq \Pi$, then $\Lambda \supseteq \lambda(\Pi)$.

(3) If Λ is *both* a π-system and a λ-system, then Λ is a σ-algebra.

int

complement.

countable union

Only NTS: $A, B \in \Lambda \Rightarrow A \cup B \in \Lambda$

$$(A \cup B) = \left(A^c \cap B^c\right)^c \in \Lambda$$

QED

(2) Write $\displaystyle\bigcup_{i=1}^{\infty} A_i = \bigcup_{n=1}^{\infty}\left(\bigcup_{i=1}^{n} A_i\right)$

$\Rightarrow$ finite unions belong $\Rightarrow$ QED.

(4) To finish the proof, we only need to show $\lambda(\Pi)$ is closed under intersections.

(5) Let $C \in \lambda(\Pi)$, and define $\Lambda_C = \{B \in \lambda(\Pi) \mid B \cap C \in \lambda(\Pi)\}$. $\boxed{\text{Then } \Lambda_C \text{ is a } \lambda\text{-system.}}$

Pf of (5):

(1) $X \in \Lambda_C$ ✓

(2) If $A_1 \subseteq A_2$, $A_i \in \Lambda_C$, NTS $A_2 - A_1 \in \Lambda_C$

i.e. NTS $(A_2 - A_1) \cap C \in \lambda(\Pi)$

$$(A_2 - A_1) \cap C = \underset{\lambda(\Pi)}{\underset{\cap}{(A_2 \cap C)}} - \underset{\lambda(\Pi)}{\underset{\cap}{(A_1 \cap C)}} \Bigg\} \in \lambda(\Pi)$$

Q.E.D.

(6) If $B, C \in \lambda(\Pi)$, then $B \cap C \in \lambda(\Pi)$.

▷ Suppose first $D \in \Pi$. Then $D \cap B \in \lambda(\Pi)$ for all $B \in \lambda(\Pi)$.

▷ For all $B \in \lambda(\Pi)$, we must have $\Lambda_B \supseteq \lambda(\Pi)$.

① $D \in \Pi$. $\quad \Lambda_D = \{ B \mid B \cap D \in \lambda(\Pi) \}$ is a λ sys.

Note $\forall B \in \Pi$, $D \cap B \in \Pi \subseteq \lambda(\Pi) \Rightarrow B \in \Lambda_D$.

$\Rightarrow \Lambda_D \supseteq \Pi \Rightarrow \Lambda_D \supseteq \lambda(\Pi)$. ($\because \Lambda_D$ is a λ-sys).

$$\boxed{\Rightarrow \forall D \in \Pi, \ D \cap B \in \lambda(\Pi) \ \forall B \in \lambda(\Pi).}$$

Next: $\forall B \in \lambda(\Pi)$.

$$\Lambda_B = \{ E \mid E \cap B \in \lambda(\Pi) \} \text{ is a } \lambda\text{-system.}$$

$$\Lambda_B \supseteq \Pi \quad (\text{by previous part})$$

$$\Rightarrow \Lambda_B \supseteq \lambda(\Pi) \Rightarrow \forall E \in \lambda(\Pi),\ E \cap B \in \lambda(\Pi)$$

$$\Rightarrow \lambda(\Pi) \text{ is a } \pi\text{-system} \Rightarrow \sigma \text{ alg} \Rightarrow \text{QED.}$$

4.2. **Regularity of measures.**

Definition 4.7. Let X be a metric space, and μ be a Borel measure on X. We say μ is regular if:

(1) For all compact sets K, we have $\mu(K) < \infty$.
(2) For all open sets U we have $\mu(U) = \sup\{\mu(K) \mid K \subseteq U \text{ is compact}\}$.
(3) For all Borel sets A we have $\mu(A) = \inf\{\mu(U) \mid U \supseteq A,\ U \text{ open}\}$.

Motivation:

- ▷ Approximation of measurable functions by continuous functions
- ▷ Differentiation of measures
- ▷ Uniqueness in the Riesz representation theorem

Question 4.8. *If μ is regular, is $\mu(A) = \sup\{\mu(K) \mid K \subseteq A,\ K \text{ compact}\}$ for all Borel sets A?*