Last time: $\lambda^* \longrightarrow$ Outer measure. ← ctably sub add (nature)

Want a σ alg $\mathcal{L}$ s.t.

$\lambda^*\big|_{\mathcal{L}}$ is a measure. ← (ctably add)

(restricted to $\mathcal{L}$)

Theorem 3.22 (Carathéodory extension). *Let* $\Sigma \overset{\text{def}}{=} \{E \subseteq X \mid \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)\ \forall A \subseteq X\}$. *Then* Σ *is a* σ*-algebra, and* μ^* *is a measure on* (X, Σ).

Remark 3.23. Clearly $\mu^*(A) \leqslant \mu^*(A \cap E) + \mu^*(A \cap E^c)$ for all E, A.

Intuition: Suppose $\mu^* = \lambda^*$. In order to show $\mu^*(A) \geqslant \mu^*(A \cap E) + \mu^*(A \cap E^c)$, cover A by cells so that $\mu^*(A) \geqslant \sum \ell(I_k) - \varepsilon$. Split this cover into cells that intersect E and cells that intersect E^c. If E is nice, hopefully the overlap is small.

Proof of Theorem 3.22

(1) $\emptyset \in \Sigma$.

(2) $E \in \Sigma \implies E^c \in \Sigma$.

(3) $E, F \in \Sigma \implies E \cup F \in \Sigma$. (Hence $E_1, \ldots, E_n \in \Sigma \implies \cup_1^n E_i \in \Sigma$.)

(4) If $E_1, \ldots, E_n \in \Sigma$ are pairwise disjoint, $A \subseteq X$, then $\mu^*(A \cap (\cup_1^n E_i)) = \sum_1^n \mu^*(A \cap E_i)$.

Pf: $n = 2$.

$\because E_1 \in \Sigma$

$$\mu^*(A \cap (E_1 \cup E_2)) = \mu^*(A \cap (E_1 \cup E_2) \cap E_1) + \mu^*(A \cap (E_1 \cup E_2) \cap E_1^c)$$

E_1, E_2 disj.

$$\overset{\text{def}}{=} \mu^*(A \cap E_1) + \mu^*(A \cap E_2)$$

QED.

(5) Σ is closed under countable *disjoint* unions, and μ^* is countably additive on Σ.

Proof: Let $E_1, E_2, \ldots, \in \Sigma$ be pairwise disjoint, and $A \subseteq X$ be arbitrary.

NTS $\bigcup_1^\infty E_i \in \Sigma$ & $\mu^*\left(\bigcup_1^\infty E_i\right) = \sum_1^\infty \mu^*(E_i)$

$\left(\because \bigcup_1^N E_i \in \Sigma\right)$

Pf: $\mu^*(A) = \mu^*\left(A \cap \bigcup_1^N E_i\right) + \mu^*\left(A \cap \left(\bigcup_1^N E_i\right)^c\right)$

$= \sum_1^N \mu^*(A \cap E_i) + \mu^*\left(A \cap \left(\bigcup_1^N E_i\right)^c\right)$

$\geq \sum_1^N \quad \text{''} \quad + \mu^*\left(A \cap \left(\bigcup_1^\infty E_i\right)^c\right)$ $\forall N$

$$\Rightarrow \mu^*(A) \geq \sum_{i=1}^{\infty} \mu(A \cap E_i) + \mu^*\Big(A \cap \big(\bigcup_{i=1}^{\infty} E_i\big)^c\Big)$$

$$\underset{\text{(ctable subadd)}}{\geq} \mu^*\Big(A \cap \big(\bigcup_{i=1}^{\infty} E_i\big)\Big) + \mu^*\Big(A \cap \big(\bigcup_{i=1}^{\infty} E_i\big)^c\Big) \geq \mu^*(A)$$

$$\Rightarrow \mu^*(A) = \mu^*\Big(A \cap \big(\bigcup_{i=1}^{\infty} E_i\big)\Big) + \mu^*\Big(A \cap \big(\bigcup_{i=1}^{\infty} E_i\big)^c\Big)$$

$$\Rightarrow \bigcup_{i=1}^{\infty} E_i \in \Sigma.$$

$\boxed{\Rightarrow \text{ctable add}}$

QED.

Remark 3.24. Note, the above shows $\mu^*(A \cap (\cup_1^\infty E_i)) = \sum_1^\infty \mu^*(A \cap E_i)$.

Definition 3.25. Define the *Lebesgue σ-algebra* by $\mathcal{L}(\mathbb{R}^d) = \{E \mid \lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c)\ \forall A \subseteq \mathbb{R}^d\}$.

Definition 3.26. Define the *Lebesgue measure* by $\lambda(E) = \lambda^*(E)$ for all $E \in \mathcal{L}(\mathbb{R}^d)$.

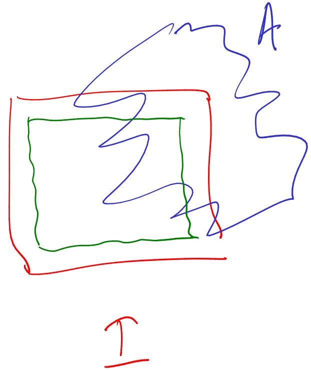
Remark 3.27. By Carathéodory, $\mathcal{L}(\mathbb{R}^d)$ is a σ-algebra, and λ is a measure on $\mathcal{L}$. (aka m)

Question 3.28. *Is $\mathcal{L}(\mathbb{R}^d)$ non-trivial?*

Yes

Proposition 3.29. *If $I \subseteq \mathbb{R}^d$ is a cell, then $I \in \mathcal{L}(\mathbb{R}^d)$.*

Proof:



NTS $\lambda^*(A) \geq \underline{\lambda^*(A \cap I) + \lambda^*(A \cap I^c)}$

Pf: Pick $\varepsilon > 0$.

$\exists\, \delta > 0$ & a cell $J_\delta \subseteq I \ni \lambda^*(I - J_\delta) < \varepsilon$

& $d(J_\delta, I^c) > 0$

Note $\lambda^*(A) \geq \lambda^*\big((A \cap J_\delta) \cup (A \cap I^c)\big)$

separated

$\underset{\text{cbl add}}{\leq}$ $\lambda^*(A \cap J_\delta) + \lambda^*(A \cap I^c)$

Note $A \cap I \subseteq (A \cap J_\delta) \cup (I - J_\delta)$

$\Rightarrow \lambda^*(A \cap I) \leq \lambda^*(A \cap J_\delta) + \varepsilon$

$\Rightarrow \lambda^*(A \cap J_\delta) \geq \lambda^*(A \cap I) - \varepsilon$

$\Rightarrow \lambda^*(A) \geq \lambda^*(A \cap I) - \varepsilon + \lambda^*(A \cap I^c)$

QED.

Proposition 3.30. $\mathcal{L}(\mathbb{R}^d) \supseteq \mathcal{B}(\mathbb{R}^d)$.

Remark 3.31. We will show later that $\mathcal{L}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R}^d) + \mathcal{N}$, where $\mathcal{N} = \{A \subseteq \mathbb{R}^d \mid \lambda^*(A) = 0\}$.

Pf: $\mathcal{L}(\mathbb{R}^d) \supseteq$ cells

$\Rightarrow \mathcal{L}(\mathbb{R}^d) \supseteq \sigma(\text{cells}) \supseteq \mathcal{B}(\mathbb{R}^d)$

(any open set is a ctable union of cells)

Here are two results that will be proved later:

Theorem 3.32. $\boxed{\mathcal{L}(\mathbb{R}^d) \supsetneq \mathcal{B}(\mathbb{R}^d).}$ *(In fact the cardinality of $\mathcal{L}(\mathbb{R}^d)$ is larger than that of $\mathcal{B}(\mathbb{R}^d)$.)*

Theorem 3.33. $\boxed{\mathcal{L}(\mathbb{R}^d) \subsetneq \mathcal{P}(\mathbb{R}^d).}$

Theorem 3.34 (Uniqueness)**.** *If μ is any measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $\mu(I) = \lambda(I)$ for all cells, then $\mu(E) = \lambda(E)$ for all $E \in \mathcal{B}(\mathbb{R}^d)$.*

Question 3.35. *Let $\mathcal{E} \subseteq \mathcal{P}(X)$, and suppose μ, ν are two measures which agree on $\mathcal{E}$. Must they agree on $\sigma(E)$?*

Pf: Claim 1: $\forall E \in \mathcal{B}(\mathbb{R}^d)$ $\mu(E) \leq \lambda(E)$.

cells

Say $E \subseteq \bigcup_1^\infty I_k \Rightarrow \mu(E) \leq \sum_1^\infty \mu(I_k) = \sum_1^\infty \lambda(I_k) = \sum_1^\infty \ell(I_k)$

$\Rightarrow \mu(E) \leq \inf\left\{ \sum \ell(I_k) \mid E \subseteq \bigcup_1^\infty I_k,\ I_k \text{ cell} \right\} = \lambda^*(E)$

$= \lambda(E)$

$\Rightarrow$ Claim 1.

Claim 2: If $E \subset \mathbb{R}^d$ is bounded, then $\mu(E) \geq \lambda(E)$.

Pf: E bdd $\Rightarrow \exists$ a cell I s.t. $I \supseteq E$

$$\mu(I - E) \overset{\text{claim 1}}{\leq} \lambda(I - E) = \cancel{\lambda(I)} - \lambda(E)$$

$$\| $$

$$\cancel{\mu(I)} - \mu(E) \qquad \Rightarrow \lambda(E) \leq \mu(E) \quad \text{QED.}$$

Claim 1 & 2 $\Rightarrow \forall E$ bdd $\mu(E) = \lambda(E)$.

$\forall E$ arb,

$$E = \bigcup_{1}^{\infty} E \cap B(0,n)$$

$$\Rightarrow \mu(E) = \lim_{n\to\infty} \mu(E \cap B(0,n))$$

$$= \lim_{n\to\infty} \lambda(E \cap B(0,n)) = \lambda(E)$$

QED.

4. **Abstract measures**

4.1. **Dynkin systems.**

Question 4.1. *Say μ, ν are two measures such that $\mu = \nu$ on $\Pi \subseteq \Sigma$. Must $\mu = \nu$ on $\sigma(\Pi)$?*

$\triangleright$ Clearly need Π to be closed under intersections.