Math 720: Homework.

Please be aware of the late homework, and academic integrity policies in the syllabus. In particular, you may collaborate, but must write up solutions on your own. You may only turn in solutions you understand. I also recommend doing (but not turning in) the optional problems. They often involve useful concepts that will come in handy as the semester progresses.

Assignment 1 (assigned 2022-08-31, due 2022-09-07).

1. Let μ be a positive measure on (X, Σ).
   (a) If $A_i \in \Sigma$ are such that $A_i \subseteq A_{i+1}$, show that $\mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{i\to\infty} \mu(A_i)$.
   (b) If $A_i \in \Sigma$ are such that $A_i \supseteq A_{i+1}$, show that $\mu(\bigcap_{i=1}^{\infty} A_i) = \lim_{i\to\infty} \mu(A_i)$, provided $\mu(A_1) < \infty$. Is it still true if $\mu(A_1) = \infty$?
2. Prove any open subset of $\mathbb{R}^d$ is a countable union of cells.
3. For each of the following sets, compute the Lebesgue outer measure.
   (a) Any countable set. (b) The Cantor set. (c) $\{x \in [0,1] \mid x \notin \mathbb{Q}\}$.
4. (a) If $V \subseteq \mathbb{R}^d$ is a subspace with $\dim(V) < d$, then show that $\lambda^*(V) = 0$.
   (b) If $P \subseteq \mathbb{R}^2$ is a polygon show that $\operatorname{area}(P) = \lambda^*(P)$.
5. Does there exist a σ-algebra whose cardinality is countably infinite? Disprove, or find an example.

Optional problems and details I omitted in class.

* Define $\mu(A)$ to be the number of elements in A. Show that μ is a measure on $(X, \mathcal{P}(X))$. (This is called the counting measure.)
* Let $x_0 \in X$ be fixed. Define $\delta_{x_0}(A) = 1$ if $x_0 \in A$ and 0 otherwise. Show that δ_{x_0} is a measure on $(X, \mathcal{P}(X))$. (This is called the delta measure at x_0.)
* Show that $\lambda^*(a + E) = \lambda^*(E)$ for all $a \in \mathbb{R}^d$, $E \subseteq \mathbb{R}^d$.
* Show that $\lambda^*(I) = \ell(I)$ for all cells. (I only proved it for closed cells in class.)
* Show that $\mathcal{B}(\mathbb{R})$ has the same cardinality as $\mathbb{R}$.
* *(Challenge)* Suppose $f_n : [0,1] \to [0,1]$ are all Riemann integrable, $0 \leqslant f_n \leqslant 1$ and $(f_n) \to 0$ pointwise. Show that $\lim\limits_{n\to\infty} \int_0^1 f_n = 0$, using only standard tools from Riemann integration.

Assignment 2 (assigned 2022-09-07, due 2022-09-14).

1. (a) Say μ is a *translation invariant* measure on $(\mathbb{R}^d, \mathcal{L})$ (i.e. $\mu(x + A) = \mu(A)$ for all $A \in \mathcal{L}$, $x \in \mathbb{R}^d$) which is finite on bounded sets. Show that $\exists c \geqslant 0$ such that $\mu(A) = c\lambda(A)$.

   (b) Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be a linear transformation, and $A \in \mathcal{L}$. Show that $T(A) \in \mathcal{L}$ and $\lambda(T(A)) = |\det(T)|\lambda(A)$.

   Hint: Express T in terms of elementary transformations.

2. (a) Let $\mathcal{E} \subseteq \mathcal{P}(X)$, and $\rho : \mathcal{E} \to [0, \infty]$ be such that $\emptyset \in \mathcal{E}$, $\rho(\emptyset) = 0$. For any $A \subseteq X$ define

$$\mu^*(A) = \inf\Big\{\sum_1^\infty \rho(E_i) \,\Big|\, E_i \in \mathcal{E}, \text{ and } A \subseteq \bigcup_1^\infty E_j\Big\}.$$

   Show that μ^* is an outer measure.

   (b) Let (X, d) be any metric space, $\delta > 0$, $\alpha \geqslant 0$ and define

$$\mathcal{E}_\delta = \{A \subseteq X \mid \operatorname{diam}(A) < \delta\} \quad \text{and} \quad \rho_\alpha(A) = \frac{\pi^{\alpha/2}}{\Gamma(1 + \frac{\alpha}{2})}\Big(\frac{\operatorname{diam}(A)}{2}\Big)^\alpha.$$

   Let $H^*_{\alpha,\delta}$ be the outer measure obtained with $\rho = \rho_\alpha$ and the collection of sets $\mathcal{E}_\delta$. Define $H^*_\alpha = \lim_{\delta\to 0} H^*_{\alpha,\delta}$. Show H^*_α is an outer measure and restricts to a measure H_α on a σ-algebra that contains all Borel sets. The measure H_α is called the *Hausdorff measure of dimension α.*

   (c) If $X = \mathbb{R}^d$, and $\alpha = d$ show that H_d is a non-zero, finite constant multiple of the Lebesgue measure. [In fact $H_d = \lambda$ because of our choice of normalization constant, but the proof requires a little more work.]

   (d) Let $S \in \mathcal{B}(X)$. Show that there exists (a unique) $d \in [0, \infty]$ such that $H_\alpha(S) = \infty$ for all $\alpha \in (0, d)$, and $H_\alpha(S) = 0$ for all $\alpha \in (d, \infty)$. This number is called the *Hausdorff dimension* of the set S.

   (e) Compute the Hausdorff dimension of the Cantor set.

3. Using notation from the previous question, let $\mathcal{S}_\delta = \{B(x, r) \mid x \in X, r \in (0, \delta)\}$. Using the collection of sets $\mathcal{S}_\delta$ and the function $\rho = \rho_\alpha$, we obtain an outer measure $S^*_{\alpha,\delta}$. As before one can show that $S^*_\alpha = \lim_{\delta\to 0} S^*_{\alpha,\delta}$ is an outer measure, and gives a Borel measure S_α.

   (a) Do there exist $c_1, c_2 \in (0, \infty)$ such that $c_1 S_\alpha(A) \leqslant H_\alpha(A) \leqslant c_2 S_\alpha(A)$ for all Borel sets A? Prove or disprove it.

   (b) If $X = \mathbb{R}^d$ with the standard metric show that $\lambda \leqslant S_d$.

   [You may assume $\rho_d(B_r) = \lambda(B_r)$. From the previous parts one can show $S_d = c\lambda$ for some finite $c > 0$. We will return to this later and show $S_d = \lambda$.]

   * *(Optional Challenge)* Show by example $S_\alpha \neq H_\alpha$ in general.

Assignment 3 (assigned 2022-09-14, due 2022-09-22).

1. Let μ, ν be two measures on (X, Σ). Suppose $\mathcal{C} \subseteq \Sigma$ is a π-system such that $\mu = \nu$ on $\mathcal{C}$.
   (a) Suppose $\exists C_i \in \mathcal{C}$ such that $\bigcup_1^\infty C_i = X$ and $\mu(C_i) = \nu(C_i) < \infty$. Show that $\mu = \nu$ on $\sigma(\mathcal{C})$.
   (b) If we drop the finiteness condition $\mu(C_i) < \infty$ is the previous subpart still true? Prove or find a counter example.
2. Let X be a metric space and μ a Borel measure on X. Suppose there exists a sequence of sets $B_n \subseteq X$ such that $\bar{B}_n \subseteq \mathring{B}_{n+1}$, $\bar{B}_n$ is compact, $X = \cup_1^\infty B_n$ and $\mu(B_n) < \infty$. Show that μ is regular. Further, for any $A \in \mathcal{B}(X)$ and $\varepsilon > 0$ show that there exists U open and C closed such that $C \subseteq A \subseteq U$ and $\mu(U - C) < \varepsilon$. If $\mu(A) < \infty$, then also show that C above can be chosen to be compact.

   [Note that the Lebesgue measure on $\mathbb{R}^d$ satisfies the assumptions in this problem, and thus for any $A \in \mathcal{L}(\mathbb{R}^d)$ and $\varepsilon > 0$ there exists U open and C closed such that $C \subseteq A \subseteq U$ and $\lambda(U - C) < \varepsilon$.]
3. (a) Find $E \in \mathcal{B}(\mathbb{R})$ so that for all $a < b$, we have $0 < \lambda(E \cap (a, b)) < b - a$.
   (b) Let $\kappa \in (0, 1/2)$. Does there exist $E \in \mathcal{B}(\mathbb{R})$ such that for all $a < b \in \mathbb{R}$, we have $\kappa(b - a) \leqslant \lambda(E \cap (a, b)) \leqslant (1 - \kappa)(b - a)$? Prove it.
4. If $A \in \mathcal{L}(\mathbb{R}^d)$ must $\lim_{x \to 0} \lambda((A + x) \cap A) = \lambda(A)$? Prove it, or find a counter example.
5. (a) Prove $\mathcal{B}(\mathbb{R}^{m+n}) = \sigma(\{A \times B \mid A \in \mathcal{B}(\mathbb{R}^m) \,\&\, B \in \mathcal{B}(\mathbb{R}^n)\})$.
   (b) Prove $\mathcal{L}(\mathbb{R}^{m+n}) \supsetneq \sigma(\{A \times B \mid A \in \mathcal{L}(\mathbb{R}^m) \,\&\, B \in \mathcal{L}(\mathbb{R}^n)\})$.
   (c) Show $\mathcal{L}(\mathbb{R}^2) \supsetneq \mathcal{B}(\mathbb{R}^2)$.

Optional problems, and details I omitted in class.

* Let $A \in \mathcal{L}(\mathbb{R}^d)$. Prove every subset of A is Lebesgue measurable $\iff \lambda(A) = 0$.
* Show that there exists $A \subseteq \mathbb{R}$ such that if $B \subseteq A$ and $B \in \mathcal{L}$ then $\lambda(B) = 0$, *and further*, if $B \subseteq A^c$ and $B \in \mathcal{L}$ then $\lambda(B) = 0$.

We say $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra if $\emptyset \in \mathcal{A}$, and $\mathcal{A}$ is closed under complements and *finite* unions. We say $\mu_0 : \mathcal{A} \to [0, \infty]$ is a (positive) *pre-measure* on $\mathcal{A}$ if $\mu_0(\emptyset) = 0$, and for any countable disjoint sequence of sets sequence $A_i \in \mathcal{A}$ *such that* $\bigcup_1^\infty A_i \in \mathcal{A}$, we have $\mu_0(\bigcup_1^\infty A_i) = \sum_1^\infty \mu_0(A_i)$.

Namely, a pre-measure is a finitely additive measure on an algebra $\mathcal{A}$, which is also countably additive for disjoint unions *that belong to the algebra.*

* *(Caratheodory extension)* If $\mathcal{A}$ is an algebra, and μ_0 is a pre-measure on $\mathcal{A}$, show that there exists a measure μ defined on $\sigma(\mathcal{A})$ that extends μ_0.
* *(An alternate approach to λ-systems.)* Let $\mathcal{M} \subseteq P(X)$. We say $\mathcal{M}$ is a *Monotone Class*, if whenever $A_i, B_i \in \mathcal{M}$ with $A_i \subseteq A_{i+1}$ and $B_i \supset B_{i+1}$ then $\bigcup_1^\infty A_i \in \mathcal{M}$ and $\bigcap_1^\infty B_i \in \mathcal{M}$. If $\mathcal{A} \subseteq P(X)$ is an algebra, then show that the *smallest* monotone class containing $\mathcal{A}$ is exactly $\sigma(A)$.

Assignment 4 (assigned 2022-09-21, due 2022-09-29).

1. Let $C \subseteq \mathbb{R}^d$ be convex. Must C be Lebesgue measurable? Must C be Borel measurable? Prove or find counter examples. [The cases $d = 1$ and $d > 1$ are different.]
2. Let $f : [0,1] \to [0,1]$ be the Cantor function, and $g(y) = \inf\{f = y\}$.
   (a) Show that f is Hölder continuous with exponent $\ln 2/\ln 3$.
   (b) Show that $g\colon [0,1] \to C$ is injective. (Here C is the Cantor set.)
3. Let (X, Σ) be a measure space, and $f, g : X \to [-\infty, \infty]$ be measurable. Suppose whenever $g = 0$, $f \neq 0$, and whenever $f = \pm\infty$, $g \in (-\infty, \infty)$. Show that $\frac{f}{g} : X \to [-\infty, \infty]$ is measurable.

   [Note that by the given data you will never get a 'meaningless' quotient of the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$. The remainder of the quotients (e.g. $\frac{1}{\infty}$) can be defined in the natural manner.]
4. Let $f_n : X \to \mathbb{R}$ be a sequence of measurable functions such that $f_n \to f$ almost everywhere (a.e.). Let $g : \mathbb{R} \to \mathbb{R}$ be a Borel function. If for a.e. $x \in X$, g is continuous at $f(x)$, then show $g \circ f_n \to g \circ f$ a.e.
5. Let (X, Σ, μ) be a measure space, and $(X, \Sigma_\mu, \bar{\mu})$ it's completion. Show that $g : X \to [-\infty, \infty]$ is Σ_μ-measurable if and only if there exists two Σ-measurable functions $f, h : X \to [-\infty, \infty]$ such that $f = h$ μ-almost everywhere, and $f \leqslant g \leqslant h$ everywhere.

Optional problems, and details I omitted in class.

* Prove that the completion Σ_μ we defined in class is the smallest μ-complete σ-algebra that contains Σ.
* Show that $f : X \to [-\infty, \infty]$ is measurable if and only if any of the following conditions hold
  (a) $\{f < a\} \in \Sigma$ for all $a \in \mathbb{R}$.
  (b) $\{f > a\} \in \Sigma$ for all $a \in \mathbb{R}$.
  (c) $\{f \leqslant a\} \in \Sigma$ for all $a \in \mathbb{R}$.
  (d) $\{f \geqslant a\} \in \Sigma$ for all $a \in \mathbb{R}$.
* Let (f_n) is a sequence of extended real valued measurable functions. Define $f(x) = \lim f_n(x)$ if the limit exists (even if the limit is $\pm\infty$), and $f(x) = 0$ otherwise. Show that f is measurable.
* Let (X, Σ, μ) be a measure space. For $A \in \mathcal{P}(X)$ define $\mu^*(A) = \inf\{\mu(E) \mid E \supset A \,\&\, E \in \Sigma\}$, and $\mu_*(A) = \sup\{\mu(E) \mid E \subseteq A \,\&\, E \in \Sigma\}$.
  (a) Show that μ^* is an outer measure.
  (b) Let $A_1, A_2, \dots \in \mathcal{P}(X)$ be disjoint. Show that $\mu_*(\bigcup_1^\infty A_i) \geqslant \sum_1^\infty \mu_*(A_i)$.
  [The set function μ_* is called an *inner measure*.]
  (c) Show that for all $A \subseteq X$, $\mu^*(A) + \mu_*(A^c) = \mu(X)$.
  (d) Let $A \subseteq \mathcal{P}(X)$ with $\mu^*(A) < \infty$. Show that $A \in \Sigma_\mu \iff \mu_*(A) = \mu^*(A)$.

Assignment 5 (assigned 2022-09-28, due 2022-10-13).

1. Let X be a metric space, and μ a regular Borel measure on X.
   (a) True or false: For any $f : X \to \mathbb{R}$ measurable and > 0 there exists $g : X \to \mathbb{R}$ continuous such that $\mu\{f \neq g\} < \varepsilon$? Prove it or find a counter example.
   (b) Do the previous subpart when $X = \mathbb{R}^d$.
2. If $f \geqslant 0$ is measurable show that $\int_X f\,d\mu = 0 \iff f = 0$ almost everywhere.
3. Let $g \geqslant 0$ be measurable, and define $\nu(A) = \int_A g\,d\mu$. Show that ν is a measure, and $\int_E f\,d\nu = \int_E fg\,d\mu$ for all $f \in \mathcal{L}^1(\nu)$. [NOTATION: We say $d\nu = g\,d\mu$.]
4. (a) Suppose $I \subseteq \mathbb{R}^d$ is a cell, and $f : I \to \mathbb{R}$ is Riemann integrable. Show that f is measurable, Lebesgue integrable and that the Lebesgue integral of f equals the Riemann integral.
   (b) Is the previous subpart true if we only assume that an improper (Riemann) integral of f exists? Prove or find a counter example.
5. For $p \in \mathbb{R}$ define define $F(y) = \displaystyle\int_0^\infty \frac{\sin(xy)}{1 + x^p}\,dx.$
   (a) For what $p \in \mathbb{R}$ is F defined? When defined, is F continuous? Prove it.
   (b) Show that F is differentiable for $p > 2$, and not differentiable when $p = 2$.
6. Let for $n \in \mathbb{N}$ define $A_n = \bigcup_{k\in\mathbb{Z}} [\frac{2k}{2^n}, \frac{2k+1}{2^n})$. If $E \in \mathcal{B}(\mathbb{R})$ does $\lim\limits_{n\to\infty} \lambda(A_n \cap E)$ exist? Prove it.

Assignment 6 (assigned 2020-09-30, due Never).

In light of your ***MIDTERM*** *this homework is optional.*

1. Let μ be the counting measure on $\mathbb{N}$, and $f : \mathbb{N} \to \mathbb{R}$ a function.
   (a) If $\sum_1^\infty |f(n)| < \infty$, then show that $\sum_{n=1}^\infty f(n) = \int_{\mathbb{N}} f\,d\mu$.
   (b) If the series $\sum_{n=1}^\infty f(n)$ is conditionally convergent, show that $\int_{\mathbb{N}} f\,d\mu$ is not defined.
2. Let X be a metric space $C \subseteq X$ be closed and $f : C \to \mathbb{R}$ be continuous.
   (a) If $0 \leqslant f \leqslant 1$, then show that there exists $F : X \to \mathbb{R}$ continuous such that $F(c) = f(c)$ for all $c \in C$. [HINT: Let $F(x) = f(x)$ for all $x \in C$, and $F(x) = \inf\{f(c) + \frac{d(x,c)}{d(x,C)} - 1 \mid c \in C\}$ for $x \notin C$.]
   (b) *(Tietze extension theorem in metric spaces)* Do the previous subpart without assuming $0 \leqslant f \leqslant 1$. [HINT: Put $g = \tan^{-1}(f)$, construct G by the previous subpart and set $F = \tan(G)$.]
3. Find a Borel measurable function $f : [0, 1] \to \mathbb{R}$ which is not continuous almost everywhere.
4. (a) Let $s, t \geqslant 0$ be two simple functions. Show directly $\int_X (s + t) = \int_X s + \int_X t$.
   (b) Let $0 \leqslant s \leqslant t$ be two simple functions. Show $\int_X s \leqslant \int_X t$.
5. Show directly $\int_X \alpha f = \alpha \int_X f$ for any $\alpha \in \mathbb{R}$ and integrable function f.

6. (a) Let $F : \mathbb{R}^d \to [0, \infty)$ be Lebesgue measurable. Show that $\int_{\mathbb{R}^d} F \, d\lambda < \infty$ if and only if for every sequence of measurable functions (f_n) such that $|f_n| \leqslant F$ almost everywhere, and (f_n) converges almost everywhere, we have $\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n \, d\lambda = \int_{\mathbb{R}^d} \lim_{n\to\infty} f_n \, d\lambda$.

   (b) Is the previous subpart true for arbitrary measure spaces?

7. (a) If f is a bounded measurable function and $\mu(X) < \infty$, then show $\int_X f d\mu = \inf\{\int_X t \, d\mu \mid t \geqslant f \text{ is simple}\}$.

   (b) If f, g are bounded measurable functions and $\mu(X) < \infty$ show directly that $\int_X (f + g) \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu$.

8. Let $f : [0, \infty) \to \mathbb{R}$ be a measurable function. We define the *Laplace Transform* of f to be the function $F(s) = \int_0^\infty \exp(-st) f(t) \, dt$ wherever defined. If f is continuous and bounded, compute $\lim_{s\to\infty} sF(s)$.

9. *(Pull back measures)* Say ν is a measure on (Y, τ) and $f : X \to Y$ is surjective.

   (a) Show that $\Sigma = \{A \subseteq X \mid f(A) \in \tau\}$ need not be a σ-algebra. If Σ is a σ-algebra, show that $\mu(A) = \nu(f(A))$ need not be a measure on (X, Σ).

   (b) Define instead $\Sigma = \{A \subseteq X \mid f^{-1}(f(A)) = A, \& f(A) \in \tau\}$, and $\mu(A) = \nu(f(A))$. Show that Σ is a σ-algebra and μ is a measure.

   (c) If $g \in L^1(Y, \nu)$, then show that $g \circ f \in L^1(X, \mu)$ and $\int_X g \circ f \, d\mu = \int_Y g \, d\nu$.

10. *(Linear change of variable)* Let $f : \mathbb{R}^d \to \mathbb{R}$ be integrable.

   (a) For any $y \in \mathbb{R}^d$ show that $\int_{\mathbb{R}^d} f(x + y) \, d\lambda(x) = \int_{\mathbb{R}^d} f(x) \, d\lambda(x)$.

   (b) If $T : \mathbb{R}^d \to \mathbb{R}^d$ an invertible linear transformation, and $E \in \mathcal{L}(\mathbb{R}^d)$. Show that

$$\int_{T^{-1}(E)} (f \circ T) |\det T| \, d\lambda = \int_E f \, d\lambda.$$

11. Show that there exists $f : \mathbb{R} \to [0, \infty)$ Borel measurable such that $\int_a^b f \, d\lambda = \infty$ for all $a, b \in \mathbb{R}$ with $a < b \in \mathbb{R}$. [Hint: Let $g(x) = \mathbf{1}_{\{|x|<1\}} |x|^{-1/2}$, and define $h(x) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} 2^{-m-n} g(x - m/n)$. Now set $f = h^2 \mathbf{1}_{\{h<\infty\}}$.]

12. Prove Hölder's inequality for $p = 1$ and $q = \infty$.

13. If $p_i, q \in [1, \infty]$ with $\sum_1^N \frac{1}{p_i} = \frac{1}{q}$, show that $\|\prod_1^n f_i\|_q \leqslant \prod \|f_i\|_{p_i}$.

14. Show that L^∞ is a Banach space.

15. For $p \in [0, 1)$ show that you need not have $\|f + g\|_p \leqslant \|f\|_p + \|g\|_p$.

16. Let $p, q \in (1, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p$ and $g \in L^q$. Show that $\int_X |fg| \, d\mu = \|f\|_p \|g\|_q$ if and only if there exists constants $\alpha, \beta \geqslant 0$ such that $\alpha f^p = \beta g^q$.

17. (a) If X is σ-finite, and $f \in L^\infty$ then show $\|f\|_\infty = \sup_{g \in L^1 - \{0\}} \frac{1}{\|g\|_1} \int_X fg \, d\mu$.

   (b) Show that the previous subpart is false if X is not σ-finite.

Assignment 7 (assigned 2022-10-12, due 2022-10-27).

1. (a) Suppose there exists $C < \infty$ such that $\int_{\mathbb{R}^d} fg \leqslant C\|f\|_p \|g\|_q$ for all $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$. Show that $\frac{1}{p} + \frac{1}{q} = 1$.

   (b) If there exists $C < \infty$ such that $\|f\|_p \leqslant C\|\nabla f\|_q$ for all $f \in C_c^\infty(\mathbb{R}^d)$, find a relation between p, q and d.

2. (a) Suppose $\varphi\colon (a,b) \to \mathbb{R}$ is strictly convex, $f\colon X \to (a,b)$ is measurable, and $\mu(X) = 1$. Find a necessary and sufficient condition on f under which $\varphi(\int_X f\,d\mu) = \int_X \varphi \circ f\,d\mu$.

   (b) Use Jensen's inequality to prove Hölder's inequality for $p \in (1, \infty)$.

   (c) Hence (or otherwise) find necessary and sufficient conditions under which equality holds in Hölder's inequality.

3. (a) Suppose $p, q, r \in [1, \infty]$ with $p < q < r$. Prove that $L^p \cap L^r \subseteq L^q$. Further, find $\theta \in (0,1)$ such that $\|f\|_q \leqslant \|f\|_p^\theta \|f\|_r^{1-\theta}$ for all $f \in L^p \cap L^q$.

   (b) If for some $p \in [1, \infty)$, $f \in L^p(X) \cap L^\infty(X)$ show that $\lim\limits_{q \to \infty} \|f\|_q = \|f\|_\infty$.

   (c) If $\mu(X) = 1$ and $f \in L^1(X)$ show $\lim\limits_{p \to 0+} \|f\|_p = \exp\bigl(\int_X \ln|f|\,d\mu\bigr)$.

4. (a) For all $p \in [1, \infty)$, show that simple functions are dense in L^p.

   (b) Let X be a metric space and μ a regular Borel measure on X. Suppose there exists a sequence of sets $B_n \subseteq X$ such that $\bar{B}_n \subseteq \mathring{B}_{n+1}$, $\bar{B}_n$ is compact and $X = \cup_1^\infty B_n$. Show that $C_c(X)$ is dense in L^p for all $p \in [1, \infty)$.

   (c) Suppose $\Sigma = \sigma(\mathcal{C})$, where $\mathcal{C} \subseteq \mathcal{P}(X)$ is countable. If μ is a σ-finite measure and $1 \leqslant p < \infty$, show that $L^p(X)$ is separable (i.e. has a countable dense subset).

   (d) What happens to the previous three subparts when $p = \infty$?

Optional problems, and details I omitted in class.

- *(Vitali Convergence Theorem Converse)* If $f_n \to f$ in L^1, show that $f_n \to f$ in measure, and $\{f_n\}$ is both uniformly integrable and tight.
- Show that dominated implies tight.
- (a) Suppose $\lim_{\lambda \to \infty} \sup_n \int_{|f_n| > \lambda} |f_n|\,d\mu = 0$. Show that there exists an increasing funciton φ with $\varphi(\lambda)/\lambda \to \infty$ as $\lambda \to \infty$, such that $\sup_n \int_X \varphi(|f_n|) < \infty$.

  (b) Suppose $\{f_n\}$ is uniformly integrable, and L^1 bounded (i.e. $\sup_n \int |f_n| < \infty$). Show that $\lim_{\lambda \to \infty} \sup_n \int_{|f_n| > \lambda} |f_n| = 0$.

  (c) Show that the previous part fails without the assumption $\sup_n \int |f_n| < \infty$.

Assignment 8 (assigned 2022-10-26, due 2022-11-03).

1. (a) If $\mu(X) < \infty$, $1 \leqslant p < q \leqslant \infty$, show $L^q(X) \subseteq L^p(X)$ and the inclusion map from $L^q(X) \to L^p(X)$ is continuous. Find an example where $L^q(X) \subsetneq L^p(X)$. [HINT: Show $\|f\|_p \leqslant \mu(X)^{\frac{1}{p}-\frac{1}{q}}\|f\|_q$.]

   (b) Let $\ell^p = L^p(\mathbb{N})$ with respect to the counting measure. If $1 \leqslant p < q$ show that $\ell^p \subsetneq \ell^q$. Is the inclusion map $\ell^p \hookrightarrow \ell^q$ continuous? Prove your answer.

2. (a) Suppose $p \in [1, \infty)$, and $f \in L^p(\mathbb{R}^d, \lambda)$. For $y \in \mathbb{R}^d$, let $\tau_y f : \mathbb{R}^d \to \mathbb{R}$ be defined by $\tau_y f(x) = f(x - y)$. Show that $(\tau_y f) \to f$ in L^p as $|y| \to 0$.

   (b) What happens for $p = \infty$?

3. Let $e_n(x) = e^{2\pi i n x}$ for $x \in \mathbb{R}$. Does does (e_n) have a subsequence that converges almost everywhere? Prove it.

4. Let $\varphi \in L^\infty(\mathbb{R})$ be such that $\varphi(x+1) = \varphi(x)$ and $\int_0^1 \varphi = 0$. If $f \in L^1(\mathbb{R})$, must $\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\varphi(nx)\,dx$ exist? Prove it, or find a counter example.

5. Let $\mathcal{M}$ be the set of all finite signed measures on (X, Σ). Show that $\mathcal{M}$ is a Banach space under the total variation norm.

Optional problems, and details I omitted in class.

* Here is an alternate approach to the Hanh and Jordan decomposition.

  (a) Let μ be a finite signed measure on (X, Σ). For $A \in \Sigma$ define $|\mu|(A)$ by

  $$\mu(A) = \sup\Big\{\sum_{B\in\pi} |\mu(B)| \;\Big|\; \pi \text{ is a finite partition of } A \text{ into measurable sets.}\Big\}$$

  Show that $|\mu|$ is a finite positive measure on X.

  (b) *(Jordan decomposition)* Show that any finite measure can be expressed uniquely as the difference of two mutually singular measures.

  (c) *(Hanh decomposition)* Show that there exists a positive set $P \in \Sigma$ such that P^c is negative.

* Show that $\|\mu_n - \mu\|_{\mathrm{TV}} \to 0$ if and only if $(\mu_n(A)) \to \mu(A)$ uniformly in A, $\forall A \in \Sigma$.

* For $p \in [1, \infty)$ define $\|f\|_{L^{p,\infty}} = \sup_{\lambda>0} \lambda\mu(\{|f| > \lambda\})^{1/p}$, and the weak L^p space (denoted by $L^{p,\infty}$) by $L^{p,\infty} = \{f \mid \|f\|_{L^{p,\infty}} < \infty\}$. [As usual, we use the convention that functions that are equal almost everywhere are identified with each other.]

  (a) If $f \in L^p$, show $f \in L^{p,\infty}$ and $\|f\|_{L^{p,\infty}} \leqslant \|f\|_p$. Is the converse true?

  (b) If $f, g \in L^{p,\infty}$, show that $f + g \in L^{p,\infty}$. Show further that $\|f+g\|_{L^{p,\infty}} \leqslant c(\|f\|_{L^{p,\infty}} + \|g\|_{L^{p,\infty}})$ for some constant c independent of f, g. [Thus $\|\cdot\|_{L^{p,\infty}}$ is called a quasi-norm, and $L^{p,\infty}$ is called a quasi-Banach space.]

  (c) If μ is σ-finite, $1 \leqslant p < q < r < \infty$ and $f \in L^{p,\infty} \cap L^{r,\infty}$ then show $f \in L^q$. [HINT: Show first that $\int_X |f|\,d\mu = \int_0^\infty \mu(|f| > t)\,dt$.]

Assignment 9 (assigned 2022-11-02, due 2022-11-10).

1. (a) For a signed measure μ, we define $\int_X f\,d\mu = \int_X f\,d\mu^+ - \int_X f\,d\mu^-$. Suppose $(f_n) \to f$, $(g_n) \to g$, and $|f_n| \leqslant g_n$ almost everywhere with respect to $|\mu|$. If $\lim \int_X g_n\,d|\mu| = \int_X g\,d|\mu| < \infty$, show that $\lim \int_X f_n\,d\mu = \int_X f\,d\mu$.

   (b) Suppose $f, f_n \in L^1$, and $(f_n) \to f$ almost everywhere. Show that

$$\lim \int |f_n - f| d|\mu| = 0 \iff \lim \int |f_n|\,d|\mu| = \int |f| d|\mu|.$$

2. Let μ, ν be two positive measures.

   (a) If $\nu(X) < \infty$ show that

$$\nu \ll \mu \iff \forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \mu(A) < \delta \implies \nu(A) < \varepsilon.$$

   [NOTE: μ is not given to be σ-finite, so you can't apply the Radon-Nikodym theorem.]

   (b) What happens without the assumption $\nu(X) < \infty$?

3. (a) Let ν_1 and ν_2 be two finite signed measures on X. Show that there exists a finite signed measure $\nu_1 \vee \nu_2$ such that $\nu_1 \vee \nu_2(A) \geqslant \nu_1(A) \vee \nu_2(A)$, and for any other finite signed measure ν such that $\nu(A) \geqslant \nu_1(A) \vee \nu_2(A)$ we ust have $\nu_1 \vee \nu_2 \leqslant \nu$.

   (b) If ν_1, ν_2 above are absolutely continuous with respect to a positive σ-finite measure μ, prove $\nu_1 \vee \nu_2 \ll \mu$ and express $\frac{d(\nu_1 \vee \nu_2)}{d\mu}$ in terms of $\frac{d\nu_1}{d\mu}$ and $\frac{d\nu_2}{d\mu}$.

4. Let $(\Omega, \mathcal{F}, \boldsymbol{P})$ be a measure space with $\boldsymbol{P}(\Omega) = 1$, and $X \in L^1(\Omega, \mathcal{F}, \boldsymbol{P})$.
   [The probabilistic interpretation is that Ω is the sample space, $A \in \mathcal{F}$ is an event, X is a random variable, and for any $B \in \mathcal{B}$, $P(X \in B)$ is the chance that $X \in B$.]

   (a) Suppose $\mathcal{G} \subseteq \mathcal{F}$ is a σ-sub-algebra of $\mathcal{F}$. Show that there exists a unique $\mathcal{G}$-measurable function, denoted by $\boldsymbol{E}(X \mid \mathcal{G})$ such that $\int_A \boldsymbol{E}(X \mid \mathcal{G})\,dP = \int_A X\,dP$ for all $A \in \mathcal{G}$. [The function $\boldsymbol{E}(X \mid \mathcal{G})$ is called the *conditional expectation* of X given $\mathcal{G}$.]

   (b) If Y is $\mathcal{G}$ measurable, show that $\boldsymbol{E}(YX \mid \mathcal{G}) = Y\boldsymbol{E}(X \mid G)$.

   (c) *(Tower property)* If $\mathcal{H} \subseteq \mathcal{G}$ is a σ-sub-algebra, show that $\boldsymbol{E}(X \mid \mathcal{H}) = \boldsymbol{E}\big(\boldsymbol{E}(X \mid \mathcal{G}) \mid \mathcal{H}\big)$ almost everywhere.

   (d) *(Conditional Jensen)* If $\varphi : \mathbb{R} \to \mathbb{R}$ is convex and $\varphi(X) \in L^1$, show that $\varphi(\boldsymbol{E}(X \mid \mathcal{G})) \leqslant \boldsymbol{E}(\varphi(X) \mid G)$ almost everywhere.

   (e) Suppose $X \in L^2(\Omega, \mathcal{F}, \boldsymbol{P})$. Show that $\boldsymbol{E}(X \mid \mathcal{G})$ is the L^2-orthogonal projection of X onto the subspace $L^2(\Omega, \mathcal{G})$. That is, show $\boldsymbol{E}(X \mid \mathcal{G}) \in L^2(\Omega, \mathcal{G})$, and $\int_\Omega (X - \boldsymbol{E}(X \mid G))Y\,dP = 0$ for all $Y \in L^2(\Omega, \mathcal{G})$.

Optional problems, and details I omitted in class.

* Show that the Radon-Nikodym theorem need not hold if μ, ν are not σ-finite.
* Show that the natural embedding from L^1 to $(L^\infty)^*$ is not necessarily surjective.

Assignment 10 (assigned 2020-11-09, due 2020-11-17).

1. (a) If X and Y are not σ-finite, show that Fubini's theorem need not hold.

   (b) If $\int_{[-1,1]^2} f\,d\lambda$ is not assumed to exist (in the extended sense), show that both iterated integrals can exist, be finite, but need not be equal.

2. *(Fubini for completions.)* Suppose (X, Σ, μ) and (Y, τ, ν) are two σ-finite, complete measure spaces. Let $\varpi = (\Sigma \otimes \tau)_\pi$ denote the completion of $\Sigma \otimes \tau$ with respect to the product measure $\pi = \mu \times \nu$.

   (a) Show that $\Sigma \otimes \tau$ need not be π-complete (i.e. $\varpi \supsetneq \Sigma \otimes \tau$ in general).

   (b) Suppose $f : X \times Y \to [-\infty, \infty]$ is ϖ-measurable. Show that for μ-almost all $x \in X$, the function $y \mapsto f(x, y)$ is τ-measurable, and for ν-almost all $y \in Y$, the function $x \mapsto f(x, y)$ is Σ-measurable.

   (c) Suppose f is integrable on $X \times Y$ in the extended sense. Define $F(x) = \int_Y f(x, y)\,d\nu(y)$ and $G(y) = \int_X f(x, y)\,d\mu(x)$. Show F is defined μ-a.e. and Σ-measurable. Similarly show G is defined ν-a.e., and τ-measurable. Further, show and that $\int_X F\,d\mu = \int_Y G\,d\nu = \int_{X\times Y} f\,d\pi$.

3. Compute $\int_0^\infty \frac{\sin x}{x}\,dx \stackrel{\text{def}}{=} \lim_{R\to\infty} \int_0^R \frac{\sin x}{x}\,dx$. [Hint: Substitute $\frac{1}{x} = \int_0^\infty e^{-xy}\,dy$.]

4. If $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p$, $g \in L^q$ show that $f * g$ is bounded and continuous. If $p, q < \infty$, show further $f * g(x) \to 0$ as $|x| \to \infty$.

5. Let $\{\varphi_n\}$ be an approximate identity.

   (a) If $f \in C_c(\mathbb{R}^d)$, show $f * \varphi_n \to f$ uniformly.

   (b) If $\varphi_n \in C_c^\infty(\mathbb{R}^d)$, $p \in [1, \infty]$, and $f \in L^p(\mathbb{R}^d)$ then show that $f * \varphi_n \in C^\infty(\mathbb{R}^d)$.

Optional problems, and details I omitted in class.

* To expand on why $(L^1)^* \subsetneq L^\infty$, we can (partially) construct a counter example as follows. The Hanh-Banach theorem shows that there exists exists $T \in (\ell^\infty)^*$ such that $Ta = \lim a_n$, for all $a = (a_n) \in \ell^\infty$ such that $\lim a_n$ exists and is finite. Show that there does not exist $b \in \ell^1$ such that $Ta = \sum a_n b_n$ for all $a \in \ell^\infty$.
* Let μ be a σ-finite measure on X, $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$, and $g \in L^1$. If for every $f \in L^p$ we know $fg \in L^1$ and further $\sup\{|\int fg| \mid \|f\|_p \leqslant 1\} = C < \infty$, then show that $g \in L^q$ and $\|g\|_q = C$. [I used this to finish the proof of the Radon-Nikodym theorem in class.]
* Show that the Lebesgue measure on $\mathbb{R}^{m+n}$ is the product of the Lebesgue measures on $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively.
* If $\{\varphi_n\}$ is an approximate identity, $\alpha \in (0, 1)$ and $f \in C_c^\alpha(\mathbb{R}^d)$, we know that $\varphi_n * f \to f$ in L^∞. Show that $\varphi_n * f$ need not converge to f in C^α.

Assignment 11 (assigned 2022-11-16, due 2022-12-01).

1. Let $A, B \in \mathcal{L}(\mathbb{R})$ be measurable, and define $A + B = \{a + b \mid a \in A, b \in B\}$. If $\lambda(A) > 0$ and $\lambda(B) > 0$ show $A + B$ contains an interval.
2. Let $\alpha \in (0, 1)$ and $f \in C^\alpha_{per}$ (i.e. f is α-Hölder continuous, and 1-periodic). Show that $(S_N f) \to f$ uniformly, as $N \to \infty$.
3. Let μ be a finite signed Borel measure on $[0, 1]$. If $\forall n \in \mathbb{Z}\ \hat{\mu}(n) = 0$, show $\mu = 0$.
4. (a) If $f \in L^1(\mathbb{R}^d)$ and f is not identically 0 (a.e.), then show that $Mf \notin L^1(\mathbb{R}^d)$.

   The next few subparts outline a proof that for any $p > 1$, the maximal function is an L^p bounded sublinear operator. Let $p \in (1, \infty)$, $f \in L^p(\mathbb{R}^d)$ and $f \geqslant 0$.

   (b) Show that $\lambda\{Mf > \alpha\} \leqslant \frac{3^d}{(1-\delta)\alpha} \int_{\{f > \delta\alpha\}} f$, for any $t > 0$, $\delta \in (0, 1)$ and $f \geqslant 0$ measurable.

   (c) Let $p \in (1, \infty]$, and $d \in \mathbb{N}$. Show that there exists a constant $c = c(p, d)$ such that $\|Mf\|_p \leqslant c\|f\|_p$ for all $f \in L^p(\mathbb{R}^d)$. [Hint: For $p < \infty$, use the previous part, the identity $\|Mf\|_p^p = \int_0^\infty p\alpha^{p-1}\lambda\{Mf > \alpha\}\, d\alpha$ and optimise in δ.]
5. (a) Suppose $f : [a, b] \to \mathbb{R}$ is a right continuous increasing function. Show that there exists a finite Borel measure μ such that $\mu((x, y]) = f(y) - f(x)$ for every $x, y \in [a, b]$. Show further that $\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{sc}} + \sum_i \alpha_i \delta_{a_i}$, where $\mu_{\mathrm{ac}} \ll \lambda$, $\alpha_i > 0$, $a_i \in [a, b)$, $\sum_i \alpha_i < \infty$, and $\mu_{\mathrm{sc}} \perp \lambda$ is such that $\mu_{\mathrm{sc}}(\{x\}) = 0$ for all $x \in \mathbb{R}$.

   (b) Let $f : [a, b] \to \mathbb{R}$ be monotone. Show that f is differentiable almost everywhere, $f' \in L^1([a, b])$ and that $|\int_a^b f'| \leqslant |f(b) - f(a)|$.
6. If $f \in L^1(\mathbb{R}^d)$, must $Mf(x) \geqslant |f(x)|$ almost everywhere? Prove it, or find a counter example.
7. Let μ be a finite measure on $\mathbb{R}^d$ that is mutually singular to the Lebesgue measure. Show that $D\mu = \infty$ almost everywhere with respect to μ.

Optional problems, and details I omitted in class.

* If $f \in L^p$, $g \in L^q$ with $p, q \in [1, \infty]$ and $1/p + 1/q \geqslant 1$, show that $f * g = g * f$.
* If $f \in L^p$, $g \in L^q$, $h \in L^r$ with $p, q, r \in [1, \infty]$ and $1/p + 1/q + 1/r \geqslant 2$, show that $(f * g) * h = f * (g * h)$.
* Define $e_n(x) = e^{2\pi i n x}$, and define the Dirichlet and Fejér kernels by $D_N = \sum_{-N}^{N} e_n$, $F_N = \frac{1}{N} \sum_0^{N-1} D_n$.

  (a) Show that $D_N(x) = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)}$. Further show $\lim_{N\to\infty} \int_\varepsilon^{1-\varepsilon} |D_N| = \infty$.

  (b) Show that $F_N(x) = \frac{\sin^2(N\pi x)}{N \sin^2(\pi x)}$, and that $\{F_N\}$ is an approximate identity.
* (a) If $f, g \in L^2_{per}([0, 1])$, show that $(f * g)^\wedge(n) = \hat{f}(n)\hat{g}(n)$.

  (b) If $f, g \in L^2_{per}([0, 1])$, show that $(fg)^\wedge(n) = \hat{f} * \hat{g}(n) \stackrel{\text{def}}{=} \sum_{m\in\mathbb{Z}} \hat{f}(m)\hat{g}(n - m)$.
* If $\alpha \in (0, 1)$, $f \in C^\alpha_{per}$, show that $\sup_{n\in\mathbb{N}} |n|^\alpha |\hat{f}(n)| < \infty$.
* Find an example of f such that $(|n|^\alpha \hat{f}(n)) \to 0$, but $f \notin C^\alpha_{per}$.

* For any $s \geqslant 0$ show that H^s_{per} is a closed subspace of L^2.
* Show that $f \in H^s_{per}$ for all $s \geqslant 0 \iff f \in C^\infty_{per}$.
* Let $f \in L^2([0,1])$. Show that there exists a unique $u \in C^\infty_{per}(\mathbb{R} \times (0,\infty))$ such that $\lim_{t\to 0^+} \|u(\cdot,t) - f(\cdot)\|_{L^2_{per}} = 0$, and $\partial_t u - \partial_x^2 u = 0$.
* Let $0 \leqslant r < s$. Show that any bounded sequence in H^s_{per} has a subsequence that is convergent in H^r_{per}.
* Let $n \in \mathbb{N} \cup \{0\}$, $\alpha \in [0,1)$ $s > 1/2 + n + \alpha$. Show that $H^s_{per} \subseteq C^{n,\alpha}_{per}[0,1]$ and the inclusion map is continuous. [Recall $C^{n,\alpha}_{per}[0,1]$ is the set of all C^n periodic functions on $\mathbb{R}$ whose n^{th} derivative is Hölder continuous with exponent α.]
* Find a function $f \in H^{1/2}_{per} - L^\infty$. [Thus, the Sobolev embedding theorem is false for $s = 1/2$.]
* Let $s \in (0,1]$ and $f \in L^2_{per}$. Show that $f \in H^s_{per}$ if and only if

$$\int_0^1 \Big(\frac{\|f - \tau_h f\|_{L^2}}{h^s}\Big)^2 \frac{dh}{h} < \infty \qquad \text{for } s < 1,$$
$$\text{and} \qquad \sup_{|h|\leqslant 1} \frac{\|f - \tau_h f\|_{L^2}}{h} < \infty \qquad \text{for } s = 1.$$

* (a) Let $n \in \mathbb{N}$ be even, $\frac{1}{n} + \frac{1}{n'} = 1$. If $\hat f \in \ell^{n'}(\mathbb{Z})$, show that $f \in L^n_{per}([0,1])$ and $\|f\|_{L^n} \leqslant \|\hat f\|_{\ell^{n'}}$. [HINT: Let $n = 2m$. Then $\|f\|^n_{L^n} = \|(f^m)^\wedge\|^2_{\ell^2}$.]

  (b) Let $s > \frac{1}{2} - \frac{1}{p} \geqslant 0$, and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in H^s_{per}$ show $\hat f \in \ell^q(\mathbb{Z})$. Further show that the map $f \mapsto \hat f$ is continuous from $H^s_{per} \to \ell^q$.

  (c) If $n \in \mathbb{N}$ is even, $s > \frac{1}{2} - \frac{1}{n}$ then show that $H^s_{per} \subseteq L^n([0,1])$ and that the inclusion map is continuous. [This is one of the Sobolev embedding theorems.]
* Let $s > 3/2$ and $f, g \in H^s_{per}$. Show that $fg \in H^1$, and further $D(fg) = (Df)g + f(Dg)$. (Here Df is the weak derivative of f.)
* Let $s \geqslant 1$, and $f \in H^s_{per}$. Show that f has a weak derivative Df, and $Df \in H^{s-1}$. Further, show that the map $f \mapsto Df : H^s \to H^{s-1}$ is linear and continuous.
* *(Infinite version of Vitali.)* Suppose $A \subseteq \cup B_\alpha$, where $\{B_\alpha\}_{\alpha\in\mathcal{A}}$ is an infinite collection of balls such that $\sup \lambda(B_\alpha) < \infty$. Show that there exists $\mathcal{A}' \subseteq \mathcal{A}$ such that the sub-collection $\{B_{\alpha'}\}_{\alpha'\in\mathcal{A}'}$ is disjoint and $A \subseteq \cup 5B_{\alpha'}$.
* Show that $f\colon [a,b] \to \mathbb{R}$ has bounded variation if and only if it is the difference of two increasing functions.

Assignment 12 (assigned 2022-11-30, due 2022-12-07).

1. Let μ be a positive finite Borel measure on $\mathbb{R}^d$, and $\alpha > 0$. Show that for every $A \subseteq \{D\mu > \alpha\}$, we must have $\mu(A) \geqslant \alpha\lambda(A)$.
2. Let $f \in L^1(\mathbb{R}^d)$. Let $S^{d-1} = \{y \in \mathbb{R}^d \mid |y| = 1\}$ be the $d-1$ dimensional sphere of radius 1. Show that there exists a unique measure σ on S^{d-1} such that

$$\int_{\mathbb{R}^d} f(x)\, dx = \int_{r\in[0,\infty)} \int_{y\in S_1} f(ry)\, r^{d-1}\, d\sigma(y)\, d\lambda(r).$$

HINT: For $A \in \mathcal{B}(S^{d-1})$ define $\sigma(A) = d \cdot \lambda(A^*)$ where $A^* = \{rx \mid x \in A, r \in [0,1]\}$. Now for any $B \in \mathcal{B}(S^{d-1})$ prove the desired equality when $f = \mathbf{1}_A$ where $A = \{rx \mid a < r < b, x \in B\}$.

3. Let $\alpha \in [0, d]$, and $A \in \mathcal{B}(\mathbb{R}^d)$. If $H_\alpha(A) < \infty$, show $\lim\limits_{r\to 0} \frac{H_\alpha(A\cap B(x,r))}{r^\alpha} = 0$ for H_α-almost all $x \notin A$.
4. If μ is a finite Borel measure on $\mathbb{R}^d$ define $\hat{\mu}(\xi) = \int e^{-2\pi i\langle x,\xi\rangle}\, d\mu(x)$. If $\hat{\mu}(\xi) = 0$ for all ξ, show that $\mu = 0$. [HINT: Show that $\int f\, d\mu = 0$ for all $f \in \mathcal{S}$.]
5. *(Lévy's continuity theorem)* Let μ, μ_n be a probability measures on $\mathbb{R}^d$ such that $(\hat{\mu}_n(\xi)) \to \hat{\mu}(\xi)$ for every $\xi \in \mathbb{R}^d$. Show that $\int_{\mathbb{R}^d} f\, d\mu_n \to \int_{\mathbb{R}^d} f\, d\mu$ for every bounded continuous function f.
HINT: First prove the theorem for $f \in \mathcal{S}$. Next show $\mu_n(|x| > 1/\lambda) \leqslant \frac{C}{\lambda^d} \int_{[-\lambda,\lambda]^d} (1-\hat{\mu}_n(\xi))\, d\xi$, and use this to prove the theorem for all bounded continuous f.
6. *(Central limit theorem)* Let $g \in L^1(\mathbb{R})$ be such that $g \geqslant 0$, $\int_{\mathbb{R}} g\, dx = 1$, $\int_{\mathbb{R}} xg(x) = 0$ and $\int_{\mathbb{R}} x^2 g(x)\, dx = \sigma^2 < \infty$. Define $g^{*n} = (g * \cdots * g)$ (n-times), and $h_n(x) = \delta_{1/\sqrt{n}} g^{*n}(x) = \sqrt{n} g^{*n}(\sqrt{n}x)$. Show

$$\hat{h}_n(\xi) \xrightarrow{n\to\infty} \exp\bigl(-2\pi^2\sigma^2\xi^2\bigr),$$

$$\text{and} \qquad \int_{\mathbb{R}} f h_n\, dx \to \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} f(x) \exp\Bigl(-\frac{x^2}{2\sigma^2}\Bigr)\, dx$$

for every bounded continuous function f.

Optional problems, and details I omitted in class.

* Let μ be a finite signed Borel measure on $\mathbb{R}^d$ such that $\mu \perp \lambda$. Show that $D|\mu| = \infty$, μ-almost everywhere, and $D|\mu| = 0$, λ-almost everywhere.
* Let $c_\alpha = \frac{\pi^{\alpha/2}}{\Gamma(1+\frac{\alpha}{2})}$ be the normalization constant from the definition of H_α, the Hausdorff measure of dimension α.
  (a) If $0 < H_\alpha(A) < \infty$, show $\limsup\limits_{r\to 0} \frac{H_\alpha(A\cap B(x,r))}{c_\alpha r^\alpha} \in [2^{-\alpha}, 1]$ for H_α-a.e. $x \in A$.
  (b) Show that there exists $\alpha < d$ and $A \subseteq \mathbb{R}^d$ with $H_\alpha(A) \in (0,\infty)$ such that

$$\liminf_{r\to 0} \frac{H_\alpha(A\cap B(x,r))}{c_\alpha r^\alpha} = 0 \quad \text{and} \quad \limsup_{r\to 0} \frac{H_\alpha(A\cap B(x,r))}{c_\alpha r^\alpha} < 1,$$

  for H^α-almost every $x \in A$.

(c) If C is the Cantor set, and $\alpha = \log 2 / \log 3$, compute $\limsup_{r \to 0} \frac{H_\alpha(C \cap B(x,r))}{c_\alpha r^\alpha}$.

* We say the family $\{E_r\}$ *shrinks nicely* to $x \in \mathbb{R}^d$ if there exists $\delta > 0$ such that for all r, $E_r \subseteq B(x,r)$ and $\lambda(E_r) > \delta\lambda(B(x,r))$. If $\{E_r\}$ shrinks nicely to x, show that $\lim \frac{1}{\lambda(E_r)} \int_{E_r} f = f(x)$ for all Lebesgue points of f.
* If $f \in L^1(\mathbb{R}^d)$, show that $Mf(x) \geqslant |f(x)|$ at all Lebesgue points of f.
* (a) If $p \in [1, \infty)$, $f \in L^p(\mathbb{R}^d)$, $g \in \mathcal{S}(\mathbb{R}^d)$, show that $f * g \in C^\infty(\mathbb{R}^d)$, and further $D^\alpha(f * g) = f * (D^\alpha g)$ for every multi-index α.

  (b) For $p \in [1, \infty)$, show that C_c^∞ and $\mathcal{S}$ are dense subsets of L^p
* Show that the mapping $f \mapsto \hat{f}$ is a bijection in the Schwartz space.
* For $f \in L^1$, the formula $\hat{f}(\xi) = \int f(x) e^{-2\pi i \langle x, \xi \rangle}$ allows us to prove many identities: E.g. $(\delta_\lambda f)^\wedge(\xi) = \hat{f}(\lambda\xi)$, etc. For $f \in L^2$, the formula $\hat{f}(\xi) = \int f(x) e^{-2\pi i \langle x, \xi \rangle}$ is no longer valid, as the integral is not defined (in the Lebesgue sense). However, most identities remain valid, and can be proved using an approximation argument. I list a couple here.

  (a) For $f \in L^1$ we know $(\tau_x f)^\wedge(\xi) = e^{-2\pi i \langle x, \xi \rangle} \hat{f}(\xi)$. Prove it for $f \in L^2$.

  (b) Similarly, show that $(\delta_\lambda f)^\wedge(\xi) = \hat{f}(\lambda\xi)$ for all $f \in L^2$.

  (c) Let $\mathcal{F}$ denote the Fourier transform operator (i.e. $Ff = \hat{f}$), and R denote the reflection operator (i.e. $Rf(x) = f(-x)$). Note that our Fourier inversion formula (for $f \in L^1$, $\hat{f} \in L^1$) is exactly equivalent to saying $\mathcal{F}^2 f = Rf$. Prove $\mathcal{F}^2 f = Rf$ for all $f \in L^2$.