Assignment 13 (assigned 2022-12-07, due never).

1. *(Uncertainty principle)* Suppose $f \in \mathcal{S}(\mathbb{R})$. Show that

$$\Big(\int_{\mathbb{R}} |xf(x)|^2 \, dx\Big)\Big(\int_{\mathbb{R}} |\xi \hat{f}(\xi)|^2 \, d\xi\Big) \geqslant \frac{1}{16\pi^2} \|f\|_{L^2}^2 \|\hat{f}\|_{L^2}^2$$

   [This illustrates a nice localisation principle about the Fourier transform. The first integral measures the spread of the function f. The second the spread of the Fourier transform $\hat{f}$. Here you show that this product is bounded below! The proof, once you know enough Physics, reduces to the above inequality.

   Hint: Consider $\int_{\mathbb{R}} x f(x) f'(x) \, dx$.]

2. *(Trace theorems)* Let $p \in \mathbb{R}^m$ be fixed. Given $f : \mathbb{R}^{m+n} \to \mathbb{R}$ define $S_p f : \mathbb{R}^n \to \mathbb{R}$ by $S_p f(y) = f(p, y)$.
   (a) Let $s > m/2$, and $s' = s - m/2$. Show that there exists a constant c such that $\|S_p f\|_{H^{s'}(\mathbb{R}^n)} \leqslant c \|f\|_{H^s(\mathbb{R}^{m+n})}$.
   (b) Show that the section operator S_p extends to a continuous linear operator from $H^s(\mathbb{R}^{m+n})$ to $H^{s'}(\mathbb{R}^n)$.
   [Given an arbitrary L^2 function on $\mathbb{R}^{m+n}$ it is of course impossible to restrict it to an m-dimensional hyper-plane. However, if your function has more than $n/2$ "Sobolev derivatives", then you can make sense of this restriction, and the restriction still has $s - n/2$ "Sobolev derivatives".]

3. Find $E \in \mathcal{L}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ such that $\lim_{r \to 0} \frac{\lambda(E \cap B(x,r))}{\lambda(B(x,r))}$ does not exist.

4. Let μ be a finite Borel measure on $\mathbb{R}^d$ such that $\mu(\{x\} = 0)$ for all $x \in \mathbb{R}^d$. True or false: For any $\alpha \in [0, \mu(\mathbb{R}^d)]$ there exists $A \in \mathcal{B}(\mathbb{R}^d)$ such that $\mu(A) = \alpha$. Prove it, or find a counter example.

5. Show that the arbitrary union of closed (non-degenerate) cubes (with sides parallel to the coordinate axis) is Lebesgue measurable. [Hint: Look up and use the Vitali covering theorem (which is stronger than the covering lemma I used). More generally one can show that the arbitrary union of convex sets with nonempty interiors is Lebesgue measurable (see Balcerzak and Kharazishvili '99).]

6. Let $p \in [1, 2]$, $f \in L^p(\mathbb{R}^d)$. Given $R > 0$ define $\hat{f}_R = (\mathbf{1}_{B(0,R)} f)^\wedge$. Is there a sequence $(R_n) \to \infty$ such that $\hat{f}_{R_n}$ converges almost everywhere? Prove it, or find a counter example.

7. Let $B = B(0,1) \subseteq \mathbb{R}^d$. Show that $|B| = \pi^{d/2}/\Gamma(1 + n/2)$, where $\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} \, dx$. [Hint: Compute $\int_{\mathbb{R}^d} e^{-|x|^2/2} \, dx$.]