# 21-720 Measure Theory.

2022-12-12

- *This is a closed book test. You may not use phones, calculators, or other electronic devices.*
- *You may not give or receive assistance.*
- *You have 3 hours. The exam has a total of 8 questions and 80 points.*
- *You may use any result from class or homework* ***PROVIDED*** *it is* independent *of the problem you want to use the result in. (You must also* ***CLEARLY*** *state the result you are using.)*

In this exam $\mathcal{L}(\mathbb{R}^d)$ denotes the Lebesgue $\sigma$-algebra on $\mathbb{R}^d$, $\mathcal{B}(X)$ denotes the Borel $\sigma$-algebra on a metric space $X$, and $\lambda$ denotes the Lebesgue measure on $\mathbb{R}^d$.

$\boxed{10}$ 1. Let $f$ be a Lebesgue measurable function such that $\int_{\mathbb{R}^d}(1+|x|)|f(x)|\,dx < \infty$. Let $\hat{f}$ denote the Fourier transform of $f$. Must $\hat{f} \in C^1$? If yes, prove it. If no, find a counter example.

$\boxed{10}$ 2. Show that any finite Borel measure on a compact metric space is regular.

$\boxed{10}$ 3. Let $\mu$ be a positive measure on $(X, \Sigma)$, and $f_n, f \in L^1(X)$ be such that $f_n \to f$ almost everywhere, and $\int_X |f_n|\,d\mu \to \int_X |f|\,d\mu$. True or false:

The family $\{f_n \mid n \in \mathbb{N}\}$ is uniformly integrable.

If true, prove it. If false, find a counter example.

$\boxed{5}$ 4. (a) Let $p, q \in [1, \infty]$ be such that $1/p + 1/q = 1$. Let $f \in L^p(\mathbb{R}^d)$, and $g \in L^q(\mathbb{R}^d)$. Show that $f * g$ continuous.

$\boxed{5}$ (b) Let $A \in \mathcal{L}(\mathbb{R}^d)$ have finite measure. Define $f(x) = \lambda(A \cap (A + x))$. Is $f$ continuous? Prove it, or find a counter example.

$\boxed{10}$ 5. Let $\mu_n$ be a sequence of finite (signed) measures on $(X, \Sigma)$ which is Cauchy under the total variation norm (i.e. for every $\varepsilon > 0$ there exists $N$ such that for every $m, n \geqslant N$ we have $\|\mu_m - \mu_n\|_{\mathrm{TV}} < \varepsilon$). Show that there exists a finite (signed) measure $\mu$ such that $\|\mu_n - \mu\|_{\mathrm{TV}} \to 0$ as $n \to \infty$.

$\boxed{10}$ 6. Prove the following special case of the Radon–Nikodym theorem: Suppose $\mu, \nu$ are two finite positive measures on $(X, \Sigma)$ and $\nu$ is absolutely continuous with respect to $\mu$. Show that there exists $f \in L^1(X, \mu)$ such that for every $A \in \Sigma$ we have $\nu(A) = \int_A f\,d\mu$.

$\boxed{10}$ 7. Let $f \in L^1(\mathbb{R}^d)$, and $Mf$ denote the maximal function. Show that there exists a constant $C$ (that does not depend on $f$) such that for every $\alpha > 0$, $\lambda\{Mf > \alpha\} \leqslant C\|f\|_{L^1}/\alpha$.

$\boxed{10}$ 8. Let $K \in C_c^\infty(\mathbb{R}^d)$ be such that $\int_{\mathbb{R}^d} K(x)\,dx = 0$. For any $\varepsilon > 0$ define $K_\varepsilon(x) = \frac{1}{\varepsilon^d} K(x/\varepsilon)$. Let $p, q \in (1, \infty)$, $\alpha \in (0, 1)$ and $f \in L^p(\mathbb{R}^d)$. (Note, $p$ and $q$ are not necessarily related.) Define $\|f\|_{B^\alpha_{p,q}}$ by

$$\|f\|_{B^\alpha_{p,q}} \stackrel{\text{def}}{=} \Big( \int_{\mathbb{R}^d} \Big( \frac{\|\tau_h f - f\|_{L^p}}{|h|^\alpha} \Big)^q \, \frac{dh}{|h|^d} \Big)^{1/q}$$

Find a constant $C$ such that for all $f \in L^p(\mathbb{R}^d)$ with $\|f\|_{B^\alpha_{p,q}} < \infty$ we have $\|f * K_\varepsilon\|_{L^p} \leqslant C\varepsilon^\alpha \|f\|_{B^\alpha_{p,q}}$. Express $C$ explicitly in terms of $K$ (e.g. write $C = \int_{\mathbb{R}^d}(1 + |h|^2)|K(h)|^3\,dh$, or something similar).