Continuous Time Finance: Midterm 2.

2023-04-05

- *This is a closed book test. You may not use phones, calculators, or other electronic devices.*
- *You may not give or receive assistance.*
- *You have 50 minutes. The exam has a total of 4 questions and 40 points.*
- *The questions are roughly ordered by difficulty. Good luck.*

In this exam W always denotes a standard Brownian motion, and the filtration $\{\mathcal{F}_t \mid t \geqslant 0\}$ is the Brownian filtration. Here are a few formulae that you can use:

- Solution formula to the Black Scholes PDE:

$$f(t,x) = \int_{-\infty}^{\infty} e^{-r\tau} g\Big(x \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)\tau + \sigma\sqrt{\tau}\, y\Big)\Big) \frac{e^{-y^2/2} dy}{\sqrt{2\pi}}\,, \qquad \tau = T - t\,.$$

- Black Scholes Formula for European calls, and the Greeks

$$c(t,x) = xN(d_+) - Ke^{-r\tau}N(d_-) \qquad d_\pm \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big)\,, \qquad N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\, dy\,,$$

$$\partial_x c = N(d_+)\,, \qquad \partial_x^2 c = \frac{1}{x\sigma\sqrt{2\pi\tau}} \exp\Big(\frac{-d_+^2}{2}\Big)\,, \qquad \partial_t c = -rKe^{-r\tau}N(d_-) - \frac{\sigma x}{\sqrt{8\pi\tau}} \exp\Big(\frac{-d_+^2}{2}\Big)\,.$$

[10] 1. Let $X_t = \int_0^t W_s\, dW_s$, and $Y_t = (1 + W_t)^2$. Compute $d[X,Y]_t$, and express your answer without using limits or integrals.

[10] 2. Consider a market with a bank and a stock. The bank has interest rate r and the stock price is modelled by a geometric Brownian motion with mean return rate α and volatility σ. A straddle option with strike K pays the holder $|S_T - K|$ at time T (here S_T is the spot price of the stock at time T). Find the arbitrage free price of this security at time $t \in [0,T]$. Express your answer in terms of $t, T, K, \alpha, \sigma, r$, the spot price of the stock S_t and the CDF of the standard normal, without using expectations or integrals.

[10] 3. Consider a market with a bank and a stock. The bank has interest rate r and the stock price is modelled by a geometric Brownian motion with mean return rate α and volatility σ. Let S_t denote the stock price at time t. Let g be a differentiable, increasing, function, and consider a security that pays $g(S_T)$ at maturity time T. Let Δ_t be the number of shares held in the replicating portfolio at time t. Must $\Delta_t \geqslant 0$? If yes, prove it. If no, find a counter example.

   Note: For full credit, you should write down a formula for Δ_t, and then either show it is nonnegative, or find a differentiable increasing function g for which you do not always have $\Delta_t \geqslant 0$.

[10] 4. Let S be a geometric Brownian motion with mean return rate α and volatility σ. Let $K, T > 0$, and suppose $S_0 > 0$ is not random. Find $\boldsymbol{E}(S_T - K)^+$ Express your final in terms of α, σ, T, K and the CDF of the standard normal, without using expectations or integrals.