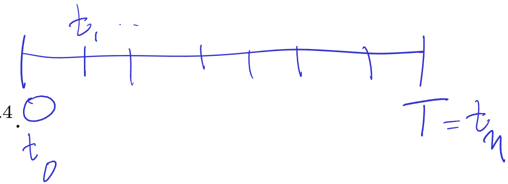
Proposition 5.8. $\boxed{[W,W]_T = T \text{ almost surely.}}$

Remark 5.9. For use in the proof: $\mathrm{Var}(\mathcal{N}(0,\sigma^2)^2) = \boldsymbol{E}\mathcal{N}(0,\sigma^2)^4 - (\boldsymbol{E}\mathcal{N}(0,\sigma^2)^2)^2 = 2\sigma^4.$



Proof:.

[illegible] $\lim_{n\to\infty} \sum (W_{t_{i+1}} - W_{t_i})^2 = T$

Proposition 5.10. $W_t^2 - [W,W]_t$ *is a martingale.*

Pf: Know from before $W_t^2 - \underline{\underline{t}}$ is a Mg.

(Computed $E_s(W_t^2 - t)$ & checked $= W_s^2 - s$)

Know $[W,W]_t = t$

$\Rightarrow W_t^2 - [W,W]_t = W_t^2 - t$ is a martingale.

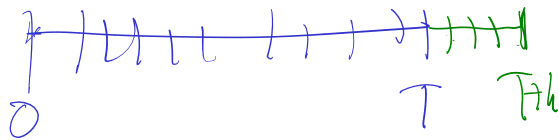
Theorem 5.11. *Let M be a continuous martingale.*

(1) $\boldsymbol{E} M_t^2 < \infty$ *if and only if* $\boldsymbol{E}[M,M]_t < \infty$.

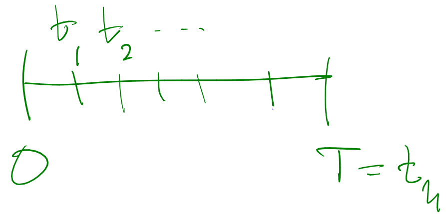
(2) *In this case* $\boxed{M_t^2 - [M,M]_t}$ *is a continuous martingale.*

(3) *Conversely, if* $M_t^2 - A_t$ *is a martingale for any continuous, increasing process* A *such that* $A_0 = 0$, *then we must have* $A_t = [M,M]_t$.

$$\left.\begin{array}{l}\text{Say } M_t^2 - A_t \text{ is a mg} \\ \text{The process } A \text{ is cts, inc \& } A_0 = 0\end{array}\right\} \Longrightarrow A_t = [M,M]_t.$$



Intuition why $M^2 - [M,M]$ is a mg



NTS $E_s\left(M_t^2 - [M,M]_t\right) = M_s^2 - [M,M]_s$.

Simpler check: Compute $E_{t_{n-1}}\left(M_{t_n}^2 - \underbrace{\sum_{i=0}^{n-1}\left(M_{t_{i+1}} - M_{t_i}\right)^2}\right)$

(Note

$$\lim_{\|P\| \to 0} \sum \left(M_{t_{i+1}} - M_{t_i}\right)^2 = [M,M]_T$$

$$E_{t_{n-1}}\left(M_{t_n}^2 - \sum_{0}^{n-1}\left(M_{t_{i+1}} - M_{t_i}\right)^2\right) = E_{t_{n-1}} M_{t_n}^2 - \underbrace{E_{t_{n-1}}\left(M_{t_n} - M_{t_{n-1}}\right)^2}$$

$$- E_{t_{n-1}}\left(\underbrace{\sum_{i=0}^{n-2}\left(M_{t_{i+1}} - M_{t_i}\right)^2}\right)$$

$\mathcal{F}_{t_{n-1}}$ meas

Compute $E_{t_{n-1}}\left(M_{t_n} - M_{t_{n-1}}\right)^2$

$$= E_{t_{n-1}}\left(M_{t_n}^2 + M_{t_{n-1}}^2 - 2M_{t_n}M_{t_{n-1}}\right)$$

$$= E_{t_{n-1}} M_{t_n}^2 + \underbrace{\cancel{E_{t_{n-1}}} M_{t_{n-1}}^2} - 2M_{t_{n-1}} E_{t_{n-1}} M_{t_n}$$

$$= E_{t_{n-1}} M_{t_n}^2 + M_{t_{n-1}}^2 - 2M_{t_{n-1}}^2 = E_{t_{n-1}} M_{t_n}^2 - M_{t_{n-1}}^2$$

$$E_{t_{n-1}}\left(M_{t_n}^2 - \sum_{0}^{n-1}\left(M_{t_{i+1}} - M_{t_i}\right)^2\right) = \cancel{E_{t_{n-1}} M_{t_n}^2} - \left(\cancel{E_{t_{n-1}} M_{t_n}^2} - M_{t_{n-1}}^2\right) - \sum_{i=0}^{n-2}\left(M_{t_{i+1}} - M_{t_i}\right)^2$$

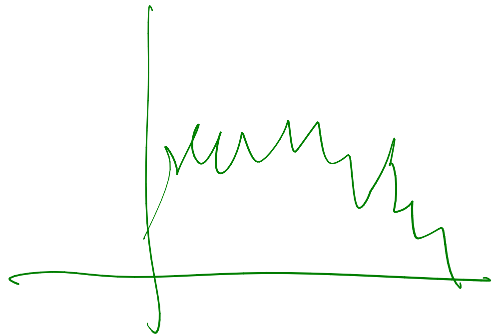
$$\Rightarrow \boxed{E_{t_{n-1}}\left(M_{t_n}^2 - \sum_{0}^{n-1}\left(M_{t_{i+1}} - M_{t_i}\right)^2\right) = M_{t_{n-1}}^2 - \sum_{i=0}^{n-2}\left(M_{t_{i+1}} - M_{t_i}\right)^2}$$

Shows a "discrete" version of $M^2 - [M,M]$ is a mg!

Remark 5.12. $\boxed{\text{If } X \text{ has finite first variation}}$, then $|X_{t+\delta t} - X_t| \approx \boxed{O(\delta t)}$.

Remark 5.13. If X has finite quadratic variation, then $|X_{t+\delta t} - X_t| \approx O(\sqrt{\delta t}) \gg O(\delta t)$.

$\delta t \longrightarrow$ small



5.4. Itô Integrals.

- $D_t = D(t)$ some adapted process (position on an asset).
- $P = \{0 = t_0 < t_1 < \cdots\}$ increasing sequence of times. (seq of trade times).
- $\|P\| = \max_i t_{i+1} - t_i$, and $\Delta_i X = X_{t_{i+1}} - X_{t_i}$.
- W : standard Brownian motion.
- $I_P(T) \stackrel{\text{def}}{=} \sum_{i=0}^{n-1} D_{t_i} \Delta_i W + D_{t_n}(W_T - W_{t_n})$

$I_P(T) =$ Itô$_D$ sum

Definition 5.14. The *Itô Integral* of D with respect to Brownian motion is defined by

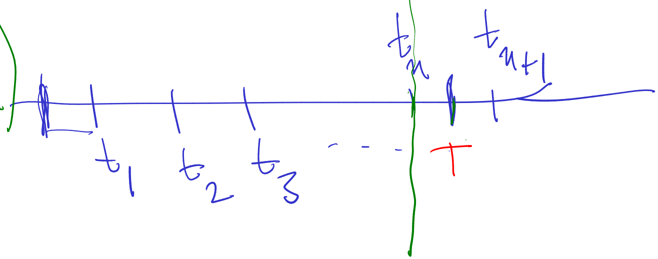
$$I_T = \int_0^T D_t \, dW_t = \lim_{\|P\| \to 0} I_P(T) \, .$$

Remark 5.15. Suppose for simplicity $T = t_n$.

(1) Riemann integrals: $\lim_{\|P\| \to 0} \sum D_{\xi_i} \Delta_i W$ exists, for any $\xi_i \in [t_i, t_{i+1}]$.

(2) Itô integrals: Need $\xi_i = t_i$ for the limit to exist.

$$I_P(T) = \underbrace{\sum_{i=0}^{n-1} D_{t_i}\left(W_{t_{i+1}} - W_{t_i}\right)} + D_{t_n}\left(W_T - W_{t_n}\right)$$



Diff btwn Ito & Riemann:

Riemann int: $\lim\limits_{\|P\|\to 0} \sum D_{t_i}(W_{t_{i+1}} - W_{t_i})$ May not exist since $V_{[0,T]} W = \infty$.

Ito int: $\lim\limits_{\|P\|\to 0} \sum D_{t_i}(W_{t_{i+1}} - W_{t_i})$ $\boxed{\text{exist with prob 1}}$ $= \int_0^T D_t \, dW_t$.

Riemann Int: Can consider $\lim\limits_{\|P\|\to 0} \sum D_{t_i}(W_{t_{i+1}} - W_{t_i})$

OR $\lim_{\|P\| \to 0} \sum D_{t_{i+1}} (W_{t_{i+1}} - W_{t_i})$

OR $\lim_{\|P\| \to 0} \sum D_{\frac{t_{i+1}+t_i}{2}} (W_{t_{i+1}} - W_{t_i})$

OR $\lim_{\|P\| \to 0} \sum D_{\xi_i} (W_{t_{i+1}} - W_{t_i})$

for any $\xi_i \in [t_i, t_{i+1}]$

For Ito:

Only Consider $\lim_{\|P\| \to 0} \sum D_{t_i} (W_{t_{i+1}} - W_{t_i})$.

Riemann Int: $\int_0^T (\ ) \, dt$ Ito Int: $\int_0^T (\ ) \, dW_t$

Theorem 5.16. *If* $\boldsymbol{E} \int_0^T D_t^2 \, dt < \infty$ *a.s., then:*

(1) $I_T = \lim_{\|P\| \to 0} I_P(T)$ *exists a.s., and* $\boldsymbol{E} I(T)^2 < \infty$.

(2) The process I_T *is a martingale:* $\boldsymbol{E}_s I_t = \boldsymbol{E}_s \int_0^t D_r \, dW_r = \int_0^s D_r \, dW_r = I_s$

(3) $[I, I]_T = \int_0^T D_t^2 \, dt$ *a.s.*

Remark 5.17. If we only had $\int_0^T D_t^2 \, dt < \infty$ a.s., then $I(T) = \lim_{\|P\| \to 0} I_P(T)$ still exists, and is finite a.s. But it may not be a martingale (it's a *local martingale*).