Math 420 Homework.

Please be aware of the late homework, and academic integrity policies in the syllabus. In particular, you may collaborate, but must write up solutions on your own. You may only turn in solutions you understand.

Assignment 1 (assigned 2023-01-18, due 2023-01-25).

1. Find the characteristic function and moment generating function of the following random variables:
   (a) A random variable that is uniformly distributed on the interval $[a, b]$.
   (b) A random variable that is exponentially distributed with parameter λ.
2. Let X be a random variable, $\alpha \in \mathbb{R}$ and set $Y = \alpha X$. Find M_Y and φ_Y in terms of M_X and φ_X.
3. Let X, Y be two independent random variables. Show $\varphi_{X+Y}(\lambda) = \varphi_X(\lambda)\varphi_Y(\lambda)$.
4. If X, Y are two random variables. The joint characteristic function of X, Y is defined by $\varphi_{X,Y}(\lambda, \mu) = \boldsymbol{E} e^{i(\lambda X + \mu Y)}$. If X, Y are independent, show that $\varphi_{X,Y}(\lambda, \mu) = \varphi_X(\lambda)\varphi_Y(\mu)$.
   [The converse is also true: Namely, if $\varphi_{X,Y}(\lambda, \mu) = \varphi_X(\lambda)\varphi_Y(\mu)$ for every $\lambda, \mu \in \mathbb{R}$, then X and Y must be independent. This however is not as easy to prove.]
5. Let $X \sim \mathcal{N}(0, 1)$, and Z be an independent random variable with $\boldsymbol{P}(Z = 1) = \boldsymbol{P}(Z = -1) = 1/2$. Let $Y = XZ$.
   (a) Find the distribution of Y. [Hint: Compute the characteristic function.]
   (b) Compute $\varphi_{X,Y}(\lambda, \mu)$ and $\varphi_X(\lambda)\varphi_Y(\mu)$.
   (c) Does $\varphi_{X+Y}(\lambda) = \varphi_X(\lambda)\varphi_Y(\lambda)$? Is this consistent with question 3.?

Assignment 2 (assigned 2023-01-25, due 2023-02-01).

1. Let X, Y be two random variables.
   (a) Show by example that if both X and Y are normally distributed and uncorrelated, then they need not be independent.
   (b) Show that if (X, Y) is jointly normal, and X and Y are uncorrelated, then X and Y are independent.
2. Let X_n be a sequence of independent random variables, and let $\mu_n = \boldsymbol{E} X_n$, $\sigma_n^2 = \operatorname{Var}(X_n)$. Suppose that as $n \to \infty$ we have $\mu_n \to \mu$ and $\sigma_n \to \sigma$. Formulate and prove the analog of the central limit theorem in this context.
3. Let W be a Brownian motion, $0 \leqslant s < t$. Show that (W_s, W_t) is jointly normal and find the covariance matrix.
4. (a) *(Chebychev's inequality)* For any $p, \lambda > 0$, prove $\boldsymbol{P}(X > \lambda) \leqslant \boldsymbol{E}(|X|^p)/\lambda^p$.
   [HINT: For $p = 1$, verify and use the fact that $\lambda \mathbf{1}_{\{X > \lambda\}} \leqslant |X|$.]
   (b) *(Jensen's inequality)* If $\varphi : \mathbb{R} \to \mathbb{R}$ is a convex function, $t \geqslant 0$, and X is a random variable, show that $\varphi(\boldsymbol{E}_t X) \leqslant \boldsymbol{E}_t \varphi(X)$.
   HINT: Use the fact that convex functions are always *above* their tangent. Namely, for any $a, x \in \mathbb{R}$, we have $\varphi(a) + (x - a)\varphi'(a) \leqslant \varphi(x)$. If this hint isn't sufficient, this should be done in most standard references.

Assignment 3 (assigned 2023-02-01, due 2023-02-08).

1. (a) If X is a continuous random variable with density p, we know $\boldsymbol{E}X = \int_{-\infty}^{\infty} xp(x)\,dx$. If X is also nonnegative, use the above formula to derive the *layer cake formula*

$$\boldsymbol{E}X = \int_0^\infty \boldsymbol{P}(X \geqslant t)\,dt\,,$$

in this special case.

(b) Let X be a nonnegative random variable (which may or may not have a density), and let φ be a differentiable, nonnegative, increasing function with $\varphi(0) = 0$. Use the layer cake formula to show that

$$\boldsymbol{E}\varphi(X) = \int_0^\infty \varphi'(t)\,\boldsymbol{P}(X \geqslant t)\,dt\,.$$

(c) Is this formula still valid if $\varphi(0) \neq 0$?

2. (a) If $s < t$, compute $\boldsymbol{E}_s W_t^4$.

(b) Given $\lambda \in \mathbb{R}$, find α so that the process $M_t = \exp(\lambda W_t - \alpha t)$ is a martingale.

3. True or false: If M is a martingale and $s < t$, then $\boldsymbol{E}(M_t - M_s)^2 = \boldsymbol{E}M_t^2 - \boldsymbol{E}M_s^2$. Prove it, or find a counter example.

4. Let Y be a standard normal random variable, and let $K \in \mathbb{R}$.

(a) For any $x \in \mathbb{R}$ let $g(x) \stackrel{\text{def}}{=} \boldsymbol{E}((e^{(x+Y)} - K)^+)$. Express g explicitly in terms of the cumulative normal distribution function

$$N(d) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{1}{2}\xi^2}\,d\xi,$$

for two different values of d. [Your answer will look something like the Black-Scholes formula.]

(b) Suppose now X is another standard normal random variable that is independent of Y. Compute $\boldsymbol{E}\big((e^{X+Y} - K)^+ \bigm| X\big)(\omega)$.

[Even though the variable ω is usually suppressed from all formulae, include it explicitly in this problem for clarity. Recall that $\boldsymbol{E}((e^{X+Y} - K)^+ \mid X)$ is shorthand for $\boldsymbol{E}((e^{X+Y} - K)^+ \mid \sigma(X))$; and $\sigma(X)$ is the smallest σ-algebra under which X is measurable. That is, when computing this conditional expectation, you can treat Y as independent of $\sigma(X)$ and X as measurable with respect to $\sigma(X)$.]

Assignment 4 (assigned 2023-02-08, due 2023-02-15).

1. Prove the tower property: If $s < t$, show that $\boldsymbol{E}_s \boldsymbol{E}_t X = \boldsymbol{E}_s X$.

   Hint: Show that $\boldsymbol{E}(\mathbf{1}_A \boldsymbol{E}_s \boldsymbol{E}_t X) = \boldsymbol{E}(\mathbf{1}_A \boldsymbol{E}_s X)$ for every $A \in \mathcal{F}_s$, without using the tower property.

2. (a) If $X \sim N(0, \sigma^2)$ find the characteristic function of X^2. (Recall if $\alpha \in \mathbb{C}$, with $\operatorname{Re}(\alpha) > 0$, then $\int_{-\infty}^{\infty} e^{-\alpha x^2} \, dx = \sqrt{\pi/\alpha}$.)

   (b) Let $T > 0$, and P be the uniform partition $P = \{0 = t_0, t_1 = T/N, t_2 = 2T/N, \dots, t_N = T\}$. Let $\xi_j = (W_{t_{j+1}} - W_{t_j})^2 - (t_{j+1} - t_j)$, and $S_N = \sum_0^{N-1} \xi_j$. Find $\varphi_{S_N}(\lambda) = \boldsymbol{E} e^{i\lambda S_N}$, and $\lim_{N\to\infty} \varphi_{S_N}(\lambda)$. [This is another way of computing the quadratic variation of Brownian motion.]

3. (a) Let $f(x) = \exp(\sin(x))$. Compute $V_{[0,T]} f$ for $T = \pi$.

   (b) Let $f(x) = x$ for $x \in [0, 1]$, and $f(x) = 2x$ for $x \in [1, 2]$. Compute $V_{[0,2]} f$.

4. (a) Let B be a continuous process with finite first variation. Show $[B, B]_T = 0$.

   (b) Use the previous part to provide an alternate proof that $V_{[0,T]}(W) = \infty$.

5. For every $p \in (0, \infty)$, we define the p-th variation of a process X by

$$V^p_{[0,T]}(X) = \lim_{\|P\|\to 0} \sum_{j=0}^{N-1} |X_{t_{i+1}} - X_{t_i}|^p \,, \text{ where } P = \{0 = t_0 < t_1 < \cdots < t_N = T\} \,.$$

   Find $\boldsymbol{E} V^p_{[0,T]}(W)$ for all $p \in (0, \infty)$.

Assignment 5 (assigned 2023-02-15, due Never).

In light of your midterm on 2022-02-22, this homework is not due. Many of the problems cover material on the midterm and are good practice. Some of the problems will be on your regular homework (due 2022-03-02).

1. Let $0 = t_0 < t_1 < t_2 \cdots$ be an increasing sequence of times, and D be an adapted process. Given $T \geqslant 0$, let $n \in \mathbb{N}$ be the unique number such that $T \in [t_n, t_{n+1})$, and define $I_P(T) = \sum_{k=0}^{n-1} D_{t_i}(W_{t_{i+1}} - W_{t_i}) + D_{t_n}(W_T - W_{t_n})$. Show that I_P is a martingale.
   [We showed in class that if $s = t_m < t_n = t$ then $\boldsymbol{E}_s I_P(t) = I_P(s)$. For this question you need to show $\boldsymbol{E}_s I_P(t) = I_P(s)$ without assuming $s = t_m$ and $t = t_n$.]
2. Let B, M be continuous adapted processes such that B has finite first variation and M is a martingale. Let $X_t = X_0 + B_t + M_t$. Show that $[X, X]_t = [M, M]_t$.
3. (a) Find functions f, g so that $W_t^4 = \int_0^t f(s, W_s)\, ds + \int_0^t g(s, W_s)\, dW_s$.

   (b) Compute $\boldsymbol{E} W_t^4$ explicitly as a function of t.

   (c) Find a function h so that $[W^4, W^4]_t = \int_0^t h(s, W_s)\, ds$.
4. Find the Itô decomposition of the process $X_t = e^{-tW_t^2}$.
5. Find a (non-random) function $g = g(t, x)$ such that the process $M_t = W_t^3 + \int_0^t g(s, W_s)\, ds$ is a martingale.
6. Compute $\boldsymbol{E}\Big[\Big(\int_0^t e^{-2s}\, dW_s\Big)^4\Big]$.
7. Let $M_t = \int_0^t s W_s\, ds$. Find $\boldsymbol{E}\big(M_t^2 - [M, M]_t\big)$.
8. Determine whether the following identities are true or false, and justify your answer.

   (a) $e^{2t} \sin(2W_t) = 2 \int_0^t e^{2s} \cos(2W_s)\, dW_s$.

   (b) $|W_t| = \int_0^t \operatorname{sign}(W_s)\, dW_s$. [Recall $\operatorname{sign}(x) = 1$ if $x > 0$, $\operatorname{sign}(x) = -1$ if $x < 0$ and $\operatorname{sign}(x) = 0$ if $x = 0$.]
9. (a) Suppose (X_1, X_2) is jointly Gaussian with $\boldsymbol{E} X_i = 0$, $\boldsymbol{E} X_i^2 = \sigma_i^2$, and $\boldsymbol{E} X_1 X_2 = \rho$. Find $\boldsymbol{E}(X_1 \mid X_2)$ (recall from your previous homework that $\boldsymbol{E}(X_1 \mid X_2)$ is shorthand for $\boldsymbol{E}(X_1 \mid \sigma(X_2))$). Express your answer in the form $g(X_2)$, where g is some function you have an explicit formula for.
   HINT: Let $Y = X_1 - \alpha X_2$, and choose $\alpha \in \mathbb{R}$ so that $\boldsymbol{E} Y X_2 = 0$. By the normal correlation theorem we know Y is independent of X_2. Now use the fact that $X_1 = Y + \alpha X_2$ to compute $\boldsymbol{E}(X_1 \mid X_2)$.

   (b) Use the previous part to compute $\boldsymbol{E}(W_s \mid W_t)$ when $s < t$. [This was asked in a job interview.]
10. Let $\alpha, \sigma \in \mathbb{R}$ and define $S_t = S_0 \exp\Big(\Big(\alpha - \frac{\sigma^2}{2}\Big)t + \sigma W_t\Big)$.

(a) Given a function $f\colon \mathbb{R} \to \mathbb{R}$, find a function $g\colon \mathbb{R} \to \mathbb{R}$ so that

$$\boldsymbol{E}\big(f(S_t) \,\big|\, \mathcal{F}_s\big) = g(S_s)\,.$$

Your formula for g will involve f and an integral involving the density of the normal distribution. [HINT: Let $Y = \exp((\alpha - \frac{\sigma^2}{2})(t-s) + \sigma(W_t - W_s))$, and note $S_t = S_s Y$ where S_s is $\mathcal{F}_s$ measurable and Y is independent of $\mathcal{F}_s$. Use this to compute $\boldsymbol{E}(f(S_s Y) \mid \mathcal{F}_s)$.]

(b) Find functions $f, g\colon \mathbb{R}^2 \to \mathbb{R}$ so that

$$S_t = S_0 + \int_0^t f(s, S_s)\,ds + \int_0^t g(s, S_s)\,dW_s\,.$$

HINT: Use the Itô formula to compute $dS_t = S_0\,d(\exp(\cdots))$. If you get the right answer you'll realize the importance of the process S to financial mathematics. The fact that I called it S and not X might have already given you a clue...

(c) Using the previous part find all $\alpha \in \mathbb{R}$ for which S is a martingale?

(d) Let $\mu_t = \boldsymbol{E} S_t$. Find a function h so that $\partial_t \mu_t = h(t, \mu_t)$. [You can do this directly using the formula for S, of course. But it might be easier (and more instructive) to use your answer to part (b) instead.]

(e) Find a function h so that $[S, S]_t = \displaystyle\int_0^t h(s, S_s)\,ds$.

In part (a) above, we observe that if we apply any function f to the process S at time t and condition it on $\mathcal{F}_s$, the whole history up to time s, we get something that only depends on S_s (the "state" at time s) and *not* anything before. This is called the Markov property. Explicitly, a process X is called *Markov* if for any function $f\colon \mathbb{R} \to \mathbb{R}$ and any $s < t$ we have $\boldsymbol{E}(f(X_t) \mid \mathcal{F}_s) = g(X_s)$ for some function g.

(f) Is Brownian motion a Markov process? Justify.

Assignment 6 (assigned 2023-02-22, due 2023-03-01).

1. Do questions 2, 6, 8, and 10 from the previous homework.

Assignment 7 (assigned 2023-03-01, due 2023-03-15).

1. Determine if the following processes are martingales.
   (a) $X(t) = (W(t) + t)\exp(-W(t) - t/2)$.
   (b) $X(t) = \Bigl(W(t) + \dfrac{t^2}{2}\Bigr)\exp\Bigl(-\displaystyle\int_0^t s\,dW(s) - \dfrac{t^3}{6}\Bigr)$
   (c) $X(t) = \Bigl(W(t) + \displaystyle\int_0^t b(s)\,ds\Bigr)\exp\Bigl(-\displaystyle\int_0^t b(s)\,dW(s) - \dfrac{1}{2}\displaystyle\int_0^t b(s)^2\,ds\Bigr)$, where b is any differentiable function of time.
2. Let $\sigma\colon [0,\infty) \to \mathbb{R}$ be a non-random function, and define $X_t = \int_0^t \sigma(s)\,dW_s$.
   (a) Show that $X_t - X_s$ is normally distributed.
   HINT: Compute the characteristic function, like we did in the proof of Lévy's theorem.
   (b) Find the mean and variance of $X_t - X_s$.
   (c) For any $\lambda \in \mathbb{R}$, compute $\boldsymbol{E}_s e^{i\lambda(X_t - X_s)}$. [HINT: Follow the proof of the first part.]
   (d) Show that $X_t - X_s$ is independent of $\mathcal{F}_s$.
3. This problem outlines how you would go about solving the Black-Scholes-Merton PDE. Suppose $f = f(t,x)$ solves $\partial_t f + rx\partial_x f + \frac{\sigma^2 x^2}{2}\partial_x^2 f = rf$, with boundary conditions $f(t,0) = 0$, linear growth as $x \to \infty$, and terminal condition $f(T,x) = g(x)$ for some given function g.
   (a) Set $y = \ln x$ and compute $\partial_x f$, $\partial_x^2 f$ in terms of y, $\partial_y f$ and $\partial_y^2 f$. Use this to find constants $\beta_1, \beta_2 \in \mathbb{R}$ such that $\partial_t f + \beta_1 \partial_y f + \beta_2 \partial_y^2 f = rf$.
   (b) Let $\tau = T - t$, $z = y + \gamma_2 \tau$ and $v(\tau, z) = e^{\gamma_1 \tau} f(t,y)$. Find γ_1 and γ_2 so that $\partial_\tau v = \kappa \partial_z^2 v$ for some constant $\kappa > 0$. Express γ_1, γ_2 and κ in terms of σ^2 and r.

   The equation you obtained for v above is called the *heat equation*, whose solution formula can be found in any standard PDE book. Namely, if we set $h(y) = v(0,y)$, then at times $\tau > 0$ the function v is given by

   $$v(\tau, y) = \frac{1}{\sqrt{4\pi\kappa\tau}} \int_{\mathbb{R}} h(y - z)\exp\Bigl(\frac{-z^2}{4\kappa\tau}\Bigr)\,dz$$

   This is very similar to the formula you should have obtained in question 10.(f).
   (c) *(Optional)* Using the above formula for v, substitute back and derive the Black, Scholes, Merton formula for c. [While this is good practice, it is a little tedious. We will derive the formula in class using risk neutral measures.]

Assignment 8 (assigned 2023-03-15, due 2023-03-22).

1. Consider a market with a bank and one stock. The bank has interest rate r and the stock price is modelled by a geometric Brownian motion with mean return rate α and volatility σ.
   (a) Consider a European call with strike K and maturity T. Let $c(t, x)$ denote the arbitrage free price of this option at time t given that the spot price of the stock is x. Use Proposition 6.8 to derive the explicit formula for c stated in Corollary 6.9.
   (b) Consider a European put with strike K and maturity T. Let $p(t, x)$ denote the arbitrage free price of this option at time t given that the spot price of the stock is x. Find a formula for p.
   HINT: You can, but need not, use Proposition 6.8.
   (c) Let $K > 0$, $a \in (0, K)$, and consider a butterfly option that matures at time T and pays $S_T - K + a$ if $S_t \in [K - a, K)$, $K + a - S_T$ if $S_T \in [K, K + a)$ and nothing otherwise. Find the arbitrage free price of this option at time $t \leqslant T$.
   (d) A digital option with strike K pays \$1 if $S_T \geqslant K$, and nothing otherwise. Find the arbitrage free price of this option at time $t \leqslant T$.
2. Consider the same market as in the previous question. Given $\gamma > 0$ a power option pays S_T^γ at maturity T. Find the arbitrage free price of this security.
   HINT: Instead of using Proposition 6.8, I suggest looking for solutions to the Black–Scholes PDE of the form $f(t, x) = \theta(t)\varphi(x)$.
3. Let $c(t, x)$ be the arbitrage free price of a European call as given by the Black–Scholes formula. Find $\lim_{t \to T^-} \partial_x c(t, x)$.

Assignment 9 (assigned 2023-03-22, due 2023-03-29).

1. *(Asian options)* Let S be a geometric Brownian motion with mean return rate α and volatility σ, modelling the price of a stock. Let $Y_t = \int_0^t S_s\,ds$.
   (a) Let $f = f(t, x, y)$ be any function that is C^2 in x, y and C^1 in t. Find a condition on f such that $X_t = f(t, S_t, Y_t)$ represents the wealth of an self financing portfolio.

   Let $g = g(x, y)$ be a function and consider a security that pays $g(S_T, Y_T)$ at time T. Note, if $g(x, y) = (y/T - K)^+$ then this is exactly an Asian option with strike price K.

   (b) Suppose this security can be replicated and $f = f(t, x, y)$ is a function such that $f(t, S_t, Y_t)$ is the wealth of the replicating portfolio of this security at time t. Assuming c is C^1 in t and C^2 in x, y when $t < T$, find a PDE and boundary conditions satisfied by c.
   [The PDE you obtain will be similar to the Black-Scholes PDE, but will also involve derivatives with respect to the new variable y. Unlike the case of European options, the PDE you obtain here will not have an explicit solution.]
   (c) Conversely, if f is the solution to the PDE you found in the previous part then show that the security can be replicated, and $f(t, S_t, Y_t)$ is the wealth of the replicating portfolio at time t.
2. Let $Y_t = \int_0^t e^{-r} W_r\,dr$, and $X_t = W_t^2 Y_t$. Find $\boldsymbol{E}_s X_t$ and $\boldsymbol{E} X_t$. Express $\boldsymbol{E}_s X_t$ in without involving expectations or conditional expectations (you may have unsimplified Itô or Riemann integrals). Express $\boldsymbol{E} X_t = h(t)$ for some (non-random) function h that you compute explicitly.
3. Let W and B be two independent (one dimensional) Brownian motions. Let M, N be defined by
$$M(t) = \int_0^t W(s)\,dB(s) \qquad \text{and} \qquad N(t) = \int_0^t B(s)\,dW(s)\,.$$
   Show $[M, N] = 0$. Also verify $\boldsymbol{E} M(t)^2 \boldsymbol{E} N(t)^2 \neq \boldsymbol{E} M(t)^2 N(t)^2$, and show that M, N are not independent even though $[M, N] = 0$.
4. Consider a market with one stock (whose price is denoted by S), and a money market account. The price of one share of the money market account is given by $C_t = e^{rt}$. At time t a self-financing portfolio holds Δ_t shares of stock and Γ_t shares of the money market account. If X_t is the wealth of this portfolio, then derive the *self-financing* condition
$$S_t\,d\Delta_t + d[S, \Delta]_t + e^{rt}\,d\Gamma_t = 0\,.$$

Assignment 10 (assigned 2023-03-29, due Never).

In light of your midterm on 2023-04-05, this homework is not due. Many of the problems cover material on the midterm and are good practice. Some of the problems will be on your regular homework (due 2023-04-12).

1. Let $X_t = \Big(W_t + \int_0^t b_s\,ds\Big)\exp\Big(-\int_0^t b_s\,dW_s - \frac{1}{2}\int_0^t b_s^2\,ds\Big)$. If b is differentiable then we have previously shown that X is a martingale. Show that X is a martingale even if b is not differentiable.
2. Let W be a two dimensional Brownian motion. Is $\displaystyle\int_0^t \frac{W_s^1}{|W_s|^2}\,dW_s^1$ a martingale? Justify.
3. Let $b = (b^1, b^2)$ be a two dimensional process, and W be a two dimensional Brownian motion.
$$Z_t = \exp\Big(-\sum_{i=1}^2 \int_0^t b_s^i\,dW_s^i - \frac{1}{2}\int_0^t |b_s|^2\,ds\Big)\,.$$
Compute dZ.
4. Let W be a two dimensional Brownian motion and $b\colon [0,\infty)\times\mathbb{R}^2 \to \mathbb{R}^2$ be a continuous function. For $i, j \in \{1,2\}$ let $\sigma_{i,j}\colon [0,\infty)\times\mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Suppose X is a stochastic process that satisfies $X_0 = x \in \mathbb{R}^2$ and
$$dX_t^i = b_i(t, X_t)\,dt + \sum_{j=1}^2 \sigma_{i,j}(t, X_t)\,dW_t^j\,.$$
Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function. Compute $\lim_{t\to 0}\frac{1}{t}(\boldsymbol{E}f(X_t) - f(x))$.
5. Let W be a two dimensional Brownian motion and define
$$B_t = \int_0^t \frac{W_s^1}{|W_s|}\,dW_s^1 + \int_0^t \frac{W_s^2}{|W_s|}\,dW_s^2\,.$$
Show that B is a Brownian motion.
6. Consider a market with two stocks S^1, S^2 and a bank with interest rate r. The stock prices are modelled by
$$dS_t^i = \alpha_i S_t^i\,dt + \sum_{j=1}^2 \sigma_{i,j} S_t^i\,dW_t^j\,,$$
where $\alpha_i, \sigma_{i,j} \in \mathbb{R}$ are constants and W is a two dimensional Brownian motion. If a security pays $g(S_T^1, S_T^2)$ at time T, show how you can price this security by solving a PDE. (You should also correctly state the boundary conditions for this PDE.)

Assignment 11 (assigned 2023-04-05, due 2023-04-12).

1. Do questions 2, 3, 5, 6 from homework 10.

Assignment 12 (assigned 2023-04-12, due 2023-04-19).

1. Consider a financial market consisting of a stock and a money market account. Suppose the money market account has a constant return rate r, and the stock price follows a geometric Brownian motion with mean return rate α and volatility σ. Here α, σ and $r > 0$ are constants. Let $\beta, K, T > 0$ and consider a derivative security that pays $(S_T^\beta - K)^+$ at maturity T. Compute the arbitrage free price of this security at any time $t \in [0, T)$. Your answer may involve r, σ, K, t, T, S, and the CDF of the normal distribution, but not any integrals or expectations.

   HINT: The simplest way to solve this problem is to use the risk neutral pricing formula, along with the explicit Black-Scholes formula you already know.

2. Consider a financial market consisting of a stock and a money market account. Suppose the money market account has a constant return rate r and the stock price is given by a stochastic process S such that

   $$dS_t = \alpha_t S_t \, dt + \sigma_t S_t \, dW_t \,.$$

   Here $\alpha = \alpha_t$ is an adapted process, and $\sigma = \sigma(t)$ is a given, non-random, function of t. Let $K, T > 0$ and consider a European call option on S with strike K and maturity T. Given $t \in [0, T)$, find the arbitrage free price of this option at time t. Your final answer may involve r, σ, t, T, K, S_t, the cumulative distribution function of the standard normal, and Riemann integrals of powers of σ.

   HINT: Under risk neutral measure find a normally distributed random variable Y such that $S_T = S_t e^Y$, and Y is independent of S_t.

3. A simplified version of the Vasiček and Ho-Lee model stipulates that the interest rate $R(t)$ is given by

   $$R(t) = r_0 + \theta t + \kappa \tilde{B}(t) \,,$$

   where r_0, $\kappa > 0$, $\theta \in \mathbb{R}$ and $\tilde{B}$ is a Brownian motion under the risk neutral measure $\tilde{\boldsymbol{P}}$. Consider a bond that pays \$1 at maturity time T. Compute the arbitrage free price of this bond at time 0. Express your answer in terms of r_0, T, θ and κ, without involving expectations or integrals.

4. Let W be a 2-dimensional Brownian motion, β, θ be two processes such that and $dX_t = \beta_t \, dt + \sin(\theta_t) \, dW_t^1 + \cos(\theta_t) \, dW_t^2$. Find infinitely many equivalent measures $\tilde{\boldsymbol{P}}$ such that X is a Brownian motion under $\tilde{\boldsymbol{P}}$. (You may assume that β is nice enough that the Girsanov theorem applies.)

Assignment 13 (assigned 2023-04-19, due 2023-04-26).

1. Let $\alpha \in \mathbb{R}^m$ and σ be a $m \times d$ matrix, and W be a d-dimensional Brownian motion. Suppose $dS^i_t = \alpha_i S^i_t \, dt + S^i_t \sum_j \sigma_{i,j} dW^j_t$.
   (a) Show that each S^i is a geometric Brownian motion. Find the mean return rate and the volatility.
   (b) Find $d[S^i_t, S^j_t]$.
   (c) Find $\boldsymbol{E}(S^i_t S^j_t)$.
2. Consider a market with a bank and one stock. The bank has interest rate r, and the stock price is modelled by $dS^1_t = \alpha S^1_t \, dt + \sigma_{1,1} S^1_t dW^1_t + \sigma_{1,2} S^1_t dW^2_t$. Here $\alpha \in \mathbb{R}$ and $\sigma_1, \sigma_2 > 0$.
   (a) Given $\alpha_2 \in \mathbb{R}$, and $\sigma_{2,1}$ and $\sigma_{2,2}$, let S^2 be a process that satisfies $dS^2_t = \alpha S^2_t \, dt + \sigma_{2,1} S^2_t dW^1_t + \sigma_{2,2} S^2_t dW^2_t$. Find $\sigma_{2,1}$ and $\sigma_{2,2}$ so that S^2 is independent of S^1.
   (b) Find infinitely many risk neutral measures. (Note the only traded stock in the market is S^1. The process S^2 is simply a process we constructed for our convenience, and does not represent the price of a traded asset.)
   (c) Explicitly find a security with maturity T and an $\mathcal{F}_T$ measurable payoff V_T so that this security can not be replicated.
3. Consider a market with 3 stocks and a bank. The bank has interest rate r, and the stock prices are modelled by $dS^i = \alpha_i S^i \, dt + \sum_{j=1}^2 \sigma_{i,j} S^i_t \, dW^j_t$, where

$$\alpha = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

   (a) Find all $r \in \mathbb{R}$ under which the market has no arbitrage.
   (b) Find all $r \in \mathbb{R}$ under which the market is complete and arbitrage free.
   (c) When the market has arbitrage, find an explicit arbitrage opportunity.

Assignment 14 (assigned 2023-04-26, due Never).

1. Let X be an Itô process such that $dX_t = t^2\,dt + t\,dW_t$. Fix $T > 0$, and let Z_T be an $\mathcal{F}_T$-measurable random variable such that $Z_T > 0$ and $\boldsymbol{E} Z_T = 1$. Define a new measure $\tilde{\boldsymbol{P}}$ by $d\tilde{\boldsymbol{P}} = Z_T\,d\boldsymbol{P}$. Find a formula for Z_T so that the process X is a martingale under $\tilde{\boldsymbol{P}}$.
2. Let B and W be two independent, standard, one dimensional Brownian motions. Compute $\boldsymbol{E} \displaystyle\int_0^{B(t)^2} W(s)^2\,ds$. Express your answer as a function of t without involving W, B, integrals, expectations or probabilities.
3. Let S be a geometric Brownian motion with mean return rate α and volatility σ. Given $T > 0$ and a non-random function f, the Markov property guarantees that there exists a non-random function g such that for any $t \leqslant T$ we have

   $$\boldsymbol{E}\big(f(S(T)) \bigm| \mathcal{F}_t\big) = g(t, S(t))\,.$$

   Find non-random functions h_1, h_2, h_3 (that may depend on x, t, α, and σ, but not on S, T, f or g) such that

   $$\partial_t g(t,x) = h_1(t,x)\partial_x g(t,x) + h_2(t,x)\partial_x^2 g(t,x) + h_3(t,x)$$

   Note: You are not required to find a formula for g itself.
4. Find an equivalent measure $\tilde{\boldsymbol{P}}$ under which $2W_t - t^2$ is a martingale. If $0 \leqslant s \leqslant t$, then compute $\tilde{\boldsymbol{E}}_s W_t$, and express your answer without using expectations or integrals.
5. Let $c(t,x)$ be given by the Black–Scholes formula. Compute $\partial_x c(t,x)$ by differentiating the risk neutral pricing formula, and use this to provide a (shorter) proof that $\partial_x c = N(d_+)$.
6. Find the forward price of a dividend paying stock. Also find the trading strategy to replicate the corresponding forward contract.

   [Recall the forward price at time t is the price at which the forward contract is worth nothing at time t.]