Continuous Time Finance: Final.

2023-05-04

- *This is a closed book test. You may not use phones, calculators, or other electronic devices.*
- *You may not give or receive assistance.*
- *You have 3 hours. The exam has a total of 8 questions and 80 points.*
- *The questions are roughly ordered by difficulty. Good luck.*

In this exam W always denotes a standard Brownian motion, and the filtration $\{\mathcal{F}_t \,|\, t \geqslant 0\}$ is the Brownian filtration. Here are a few formulae that you can use:

- Solution formula to the Black Scholes PDE:

$$f(t,x) = \int_{-\infty}^{\infty} e^{-r\tau} g\Big(x \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)\tau + \sigma\sqrt{\tau}\, y\Big)\Big) \frac{e^{-y^2/2} dy}{\sqrt{2\pi}}\,, \qquad \tau = T - t\,.$$

- Black Scholes Formula for European calls, and the Greeks

$$c(t,x) = xN(d_+) - Ke^{-r\tau}N(d_-) \qquad d_\pm \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big)\,, \qquad N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\, dy\,,$$

$$\partial_x c = N(d_+)\,, \qquad \partial_x^2 c = \frac{1}{x\sigma\sqrt{2\pi\tau}} \exp\Big(\frac{-d_+^2}{2}\Big)\,, \qquad \partial_t c = -rKe^{-r\tau}N(d_-) - \frac{\sigma x}{\sqrt{8\pi\tau}} \exp\Big(\frac{-d_+^2}{2}\Big)\,.$$

[10] 1. Express tW_t^3 as the sum of a martingale and a process of finite first variation.

[10] 2. Let W and B be two independent, standard, one dimensional, Brownian motions. Let $X_t = W_t^2 B_t$. Find the quadratic variation of X, and express your answer in the form $\int_0^t f(s, W_s, B_s)\, ds$ for some function f that you find an explicit formula for.

[10] 3. Let $\lambda \in \mathbb{R}$, and $0 < s < t$. Compute $\boldsymbol{E}_s e^{i\lambda W_t}$, where $i = \sqrt{-1}$. Your final answer should not involve any expectations or integrals.

[10] 4. Consider a market with one stock and a bank. The bank has interest rate $r \geqslant 0$ and the stock price (denoted by S_t) is governed by the equation

$$dS_t = \alpha S_t\, dt + \sigma\sqrt{S_t}\, dW_t\,,$$

for some (known) constants $\alpha, \sigma > 0$. Suppose $f = f(t,x)$ is some function such that $X_t = f(t, S_t)$ is the wealth of a self financing portfolio at time t. Find a PDE satisfied by f (i.e. express $\partial_t f$ in terms of f, $\partial_x f$, $\partial_x^2 f$ and the model parameters α, σ, r).

[10] 5. Consider a dividend paying stock whose price is modelled by

$$dS_t = \alpha_t S_t\, dt + \sigma_t S_t dW_t - A_t S_t\, dt\,.$$

Here α_t is the mean return rate, σ_t the volatility, and A_t is the rate at which the stock pays dividends. If an investor buys one share of this stock at time 0, and re-invests all dividends in the stock, then how many shares do they have at time T? (Your final answer must be an explicit formula, but may use unsimplified integrals involving σ, α and A.)

[10] 6. Consider a market with 2 stocks and a bank. The bank has interest rate $r \geqslant 0$, and the stock prices (denoted by S_t^1 and S_t^2, respectively) are modelled by

$$dS_t^1 = 2S_t^1\, dt + 2S_t^1\, dW_t^1 + 4S_t^1\, dW_t^2\,, \qquad dS_t^2 = S_t^2\, dt + 3S_t^2 dW_t^1 + 6S_t^2\, dW_t^2\,,$$

(The stocks do not pay dividends.) Does there exist $r \geqslant 0$ such that the market is complete and arbitrage free? Prove your answer. (If your answer is yes, also find all such r.)

10 7. Consider a financial market consisting of a stock and a money market account. Suppose the money market account has a constant return rate r, and the stock price follows a geometric Brownian motion with mean return rate α and volatility σ. Here α, σ and $r > 0$ are constants. Let $\beta, K, T > 0$ and consider a derivative security that pays $(S_T^\beta - K)^+$ at maturity T. Compute the arbitrage free price of this security at any time $t \in [0, T)$. Your answer may involve r, σ, K, t, T, S, and the CDF of the normal distribution, but not any integrals or expectations.

10 8. Suppose W is a 2 dimensional Brownian motion, $\alpha \in \mathbb{R}^2$ and $\sigma = (\sigma^{i,j})$ is a 2×2 matrix. (Both α and σ are not random, and independent of time.) For $i \in \{1, 2\}$, define S^i by

$$dS^i = \alpha^i S_t^i \, dt + S^i \sum_{j=1}^{2} \sigma^{i,j} \, dW_t^j \, .$$

Let $X_t = S_t^1 S_t^2$. Find $\tilde{\alpha}, \tilde{\sigma} \in \mathbb{R}$ and a one dimensional Brownian motion $\tilde{W}$ so that

$$dX_t = \tilde{\alpha} X_t \, dt + \tilde{\sigma} X_t \, d\tilde{W}_t \, .$$