12. **Review problems**

Problem 12.1. Consider the N period Binomial model with $N = 5$, and parameters $0 < d < 1 + r < u$. At maturity $N = 5$, a security pays \$1 if $S_5 > (1 + r)S_4$, and 0 otherwise. Find the arbitrage free price and trading strategy trading at time 0.

Problem 12.2. Let f be a deterministic function, and define

$$X_t \stackrel{\text{def}}{=} \int_0^t f(s) W_s \, ds \,.$$

Find the distribution of X.

Problem 12.3. Suppose σ, τ, ρ are three deterministic functions and M and N are two continuous martingales with respect to a common filtration $\{\mathcal{F}_t\}$ such that $M_0 = N_0 = 0$, and

$$d[M,M]_t = \sigma_t \, dt \,, \quad d[N,N]_t = \tau_t \, dt \,, \qquad \text{and} \qquad d[M,N]_t = \rho_t \, dt \,.$$

(a) Compute the joint moment generating function $\boldsymbol{E} \exp(\lambda M(t) + \mu N(t))$.
(b) *(Lévy's criterion)* If $\sigma = \tau = 1$ and $\rho = 0$, show that (M, N) is a two dimensional Brownian motion.

Problem 12.4. Consider a financial market consisting of a risky asset and a money market account. Suppose the return rate on the money market account is r, and the price of the risky asset, denoted by S, is a geometric Brownian motion with mean return rate α and volatility σ. Here r, α and σ are all deterministic constants. Compute the arbitrage free price of derivative security that pays

$$V_T = \frac{1}{T} \int_0^T S_t \, dt$$

at maturity T. Also compute the trading strategy in the replicating portfolio.

Problem 12.5. Let $X \sim N(0,1)$, and $a, \alpha, \beta \in \mathbb{R}$. Define a new measure $\tilde{\boldsymbol{P}}$ by

$$d\tilde{\boldsymbol{P}} = \exp\big(\alpha X + \beta\big)\, d\boldsymbol{P}\,.$$

Find α, β such that $X + a \sim N(0,1)$ under $\tilde{\boldsymbol{P}}$.

Problem 12.6. Let $x_0, \mu, \theta, \sigma \in \mathbb{R}$, and suppose X is an Itô process that satisfies

$$dX(t) = \theta(\mu - X_t)\,dt + \sigma\,dW_t\,,$$

with $X_0 = x_0$.

(a) Find functions $f = f(t)$ and $g = g(s,t)$ such that

$$X(t) = f(t) + \int_0^t g(s,t)\,dW_s\,.$$

The functions f, g may depend on the parameters x_0, θ, μ and σ, but should not depend on X.

(b) Compute $\boldsymbol{E} X_t$ and $\operatorname{cov}(X_s, X_t)$ explicitly.

Problem 12.7. Let $\theta \in \mathbb{R}$ and define

$$Z(t) = \exp\Big(\theta W_t - \frac{\theta^2 t}{2}\Big)\,.$$

Given $0 \leqslant s < t$, and a function f, find a function such that

$$\boldsymbol{E}_s f(Z_t) = g(Z(s))\,.$$

Your formula for the function g can involve f, s, t and integrals, but not the process Z or expectations.

Problem 12.8. Let W be a Brownian motion, and define

$$B_t = \int_0^t \operatorname{sign}(W_s)\, dW_s\,.$$

(a) Show that B is a Brownian motion.

(b) Is there an adapted process σ such that

$$W_t = \int_0^t \sigma_s\, dB_s\,?$$

If yes, find it. If no, explain why.

(c) Compute the joint quadratic variation $[B, W]$.

(d) Are B and W uncorrelated? Are they independent? Justify.

Problem 12.9. Let W be a Brownian motion. Does there exist an equivalent measure $\tilde{\boldsymbol{P}}$ under which the process tW_t is a Brownian motion? Prove it.