LECTURE NOTES ON STOCHASTIC CALCULUS FOR FINANCE
FALL 2021

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Contents

Note: The page numbers and links will not be correct in the annotated version.

1. **Preface.**

These are the slides I used while teaching this course in Fall 2021. I projected them (spaced out) in class, and filled in the proofs by writing over them. The annotated version of these slides with handwritten proofs, blank slides (so you take notes), and the compactified un-annotated version for quick review can be found on the class website. The LaTeXsource of these slides is also available on git.

2. Syllabus Overview

- Class website and full syllabus: https://www.math.cmu.edu/~gautam/sj/teaching/2021-22/944-scalc-finance1
- TA's: Shukun Long <shukunl@andrew.cmu.edu>.
- Homework Due: 10:10AM Oct 28, Nov 4, 11, 23, 30, Dec 7
- Midterm: Tue, Nov 16, in class (May be delayed to Nov 18 if we have not covered Itô's formula in time.)
- **Homework:**
  - ▷ Good quality scans please! Use a scanning app, and not simply take photos. (I use Adobe Scan.)
  - ▷ 20% penalty if turned in within an hour of the deadline. 100% penalty after that.
  - ▷ One homework assignments can be turned in 24h late without penalty.
  - ▷ Bottom homework score is dropped from your grade (personal emergencies, interviews, other deadlines, etc.).
  - ▷ Collaboration is encouraged. Homework is not a test – ensure you learn from doing the homework.
  - ▷ You must write solutions independently, and can only turn in solutions you fully understand.
- **Academic Integrity**
  - ▷ Zero tolerance for violations (automatic **R**).
  - ▷ Violations include:
    - Not writing up solutions independently and/or plagiarizing solutions
    - Turning in solutions you do not understand.
    - Seeking, receiving or providing assistance during an exam.
  - ▷ All violations will be reported to the university, and they may impose additional penalties.
- **Grading:** 10% homework, 30% midterm, 60% final.

Course Outline.

- Review of Fundamentals: Replication, arbitrage free pricing.
- Quick study of the multi-period binomial model.
  - ▷ Simple example of replication / arbitrage free pricing.
  - ▷ Understand conditional expectations. (Have an explicit formula.)
  - ▷ Understand measurablity / adaptedness. (Can be stated easily in terms of coin tosses that have / have not occurred.)
  - ▷ Understand risk neutral measures. Explicit formula!
- Develop tools to price securities in continuous time.
  - ▷ Brownian motion (not as easy as coin tosses)
  - ▷ Conditional expectation: No explicit formula!
  - ▷ Itô formula: main tool used for computation. Develop some intuition.
  - ▷ Measurablity / risk neutral measures: much more abstract. Complete description is technical. But we need a working knowledge.
  - ▷ Derive and understand the Black-Scholes formula.

3. Replication and Arbitrage

3.1. Replication and arbitrage free pricing.

- Start with a *financial market* consisting of traded assets (stocks, bonds, money market, options, etc.)
- We model the price of these assets through random variables (stochastic processes).
- **No Arbitrage Assumption:**
  - ▷ In order to make money, you have to take risk. (Can't make something out of nothing.)
  - ▷ Mathematically: For any trading strategy such that $X_0 = 0$, and $X_n \geqslant 0$, you must also have $X_n = 0$ almost surely.
  - ▷ Equivalently: There doesn't exist a trading strategy with $X_0 = 0$, $X_n \geqslant 0$ and $\boldsymbol{P}(X_n > 0) > 0$.
- Now consider a non-traded asset Y (e.g. an option). How do you price it?
- *Arbitrage free price:* If given the opportunity to trade Y at price V_0, the market remains arbitrage free, then we say V_0 is the arbitrage free price of Y.

Bank ← interest rate r.

Stocks/Bonds

etc.

Market

No Arb: To make \$ you have to take risk.

If

Say $X_0 = 0$ & $X_n \geqslant 0$

X_0 ~ initial wealth.

X_n ~ Wealth at time n.

Then: $X_n = 0$

Arbitrage free Price.

Market.

$Y \rightarrow$ Non traded asset.
(e.g. Call option).

AFP: If given the opportunity to trade the NT asset at price V_0 the market remains arb free, then we call V_0 = the arb free price.

- *We will almost always find the arbitrage free price by* ***replication***.
  - ▹ Say the non-traded asset pays V_N at time N (e.g. call options).
  - ▹ Try and *replicate the payoff*:
    - Start with X_0 dollars.
    - Use only traded assets and ensure that at maturity $X_N = V_N$.
  - ▹ Then the arbitrage free price is uniquely determined, and must be X_0.

Remark 3.1. The arbitrage free price is *unique* if and only if there is a replicating strategy! In this case, the arbitrage free price is exactly the initial capital of the replicating strategy.

Find AFP by $\boxed{\text{Replication.}}$

$V_N \longrightarrow$ Payoff at time N.

Replication : ① Start with X_0 \$.

② Use only traded assets.

③ Goal End with $\underbrace{X_N}_{\text{Wealth at time } N.} = V_N.$

Replicate a Security.

If you replicate a security with some trading strat.

X_0 = initial wealth

X_1 = wealth at time 1

$\vdots$

X_N = " " " N. $\left(X_N \overset{\text{(Replication)}}{=} V_N \right)$.

V_N: payoff.

$$\boxed{\text{Then AFP of the security must be } X_0.}$$

3.2. **Example: One period Binomial model.**

- Consider a market with a stock, and money market account.
- Interest rate for borrowing and lending is r. No transaction costs. Can buy and sell fractional quantities of the stock.
- *Model assumption:* Flip a coin that lands heads with probability $p_1 \in (0,1)$ and tails with probability $q_1 = 1 - p_1$. Model $S_1 = uS_0$ if heads, and $S_1 = dS_0$ if tails.
  - ▷ S_0 is stock price at time 0 (known).
  - ▷ S_1 is stock price after one time period (random).
  - ▷ u, d are model parameters (pre-supposed). Called the up and down factors. (Will always assume $0 < d < u$.)

Proposition 3.2. *There's no arbitrage in this model if and only if $d < 1 + r < u$.*

Proof:

$u, d, p, q \to$ Model parameters.

M.M $\leftarrow$ interest rate $r > -1$

Stock:

Market.

Model assumption.

S_0 = price at time 0 of stock.

$$\begin{tikzcd}[row sep=small] & uS_0 \\ S_0 \arrow[ur, "p"] \arrow[dr, "q"'] & \\ & dS_0 \end{tikzcd}$$

S_1

$$S_1 = \begin{cases} uS_0 & \text{if heads.} \\ dS_0 & \text{if tails.} \end{cases}$$

Remark: No arb in this model

$\iff d < 1+r < u.$

Intuition: ① If $\underbrace{d \geq 1+r}$. $\longrightarrow$ Bank: $X_0 \longrightarrow (1+r)X_0$.

Stock: $S_0 \longrightarrow \underbrace{\begin{cases} uS_0 \\ dS_0 \end{cases}}_{\geq (1+r)S_0}$

Arb Opportunity!

② If $1+r \geq u \rightarrow$ Reverse.

③ Calc & check that

$d < 1+r < u \Rightarrow$ No arb.

Proposition 3.3. *Say a security pays V_1 at time 1 (V_1 can depend on whether the coin flip is heads or tails). The arbitrage free price at time 0 is given by*

$$\longrightarrow \quad V_0 = \frac{1}{1+r}\big(\tilde{p}_1 V_1(H) + \tilde{q}_1 V_1(T)\big) = \frac{1}{1+r}\tilde{\boldsymbol{E}} V_1\,, \qquad \textit{where } \tilde{p}_1 = \frac{1+r-d}{u-d}\,, \quad \tilde{q} = \frac{u-(1+r)}{u-d}\,.$$

The replicating strategy holds $\Delta_0 = \underline{\dfrac{V_1(H) - V_1(T)}{(u-d)S_0}}$ *shares of stock at time 0.*

Proof:

(Assume no arb).

Security pay V_1 at time 1

($V_1 \to$ can depend on outcome of first coin toss.)

Claim: AFP at time 0: $V_0 = \dfrac{1}{1+r}\left(\underline{\tilde{p}}\, V_1(H) + \underline{\tilde{q}}\, V_1(T)\right)$.

$$\tilde{p} = \frac{1+r-d}{u-d} \qquad \& \quad \tilde{q} = \frac{u-(1+r)}{u-d}.$$

Reasons: Try & replicate $\underbrace{V_1}$

Start with X_0 \$ → cash. $(X_0 - \Delta_0 S_0)$.
→ Stock. Δ_0 shares.

X_1 = wealth at time $1 = \underbrace{\Delta_0}_{\text{\# shares at time 0}} \underbrace{S_1}_{\text{new price of stock}} + (1+r)(X_0 - \Delta_0 S_0)$.

Replication: Want $X_1 = V_1$ weather heads or tails!

$$V_1(H) = X_1(H) = \Delta_0 (u S_0) + (1+r)(X_0 - \Delta_0 S_0) \quad \text{(if heads)}.$$

$$V_1(H) = X_1(T) = \Delta_0 (d S_0) + (1+r)(X_0 - \Delta_0 S_0) \quad \text{(if tails)}.$$

2 Eqns. (linear)

2 Unknowns. $(X_0 \ \& \ \Delta_0)$. } Solve → Gives the formula.

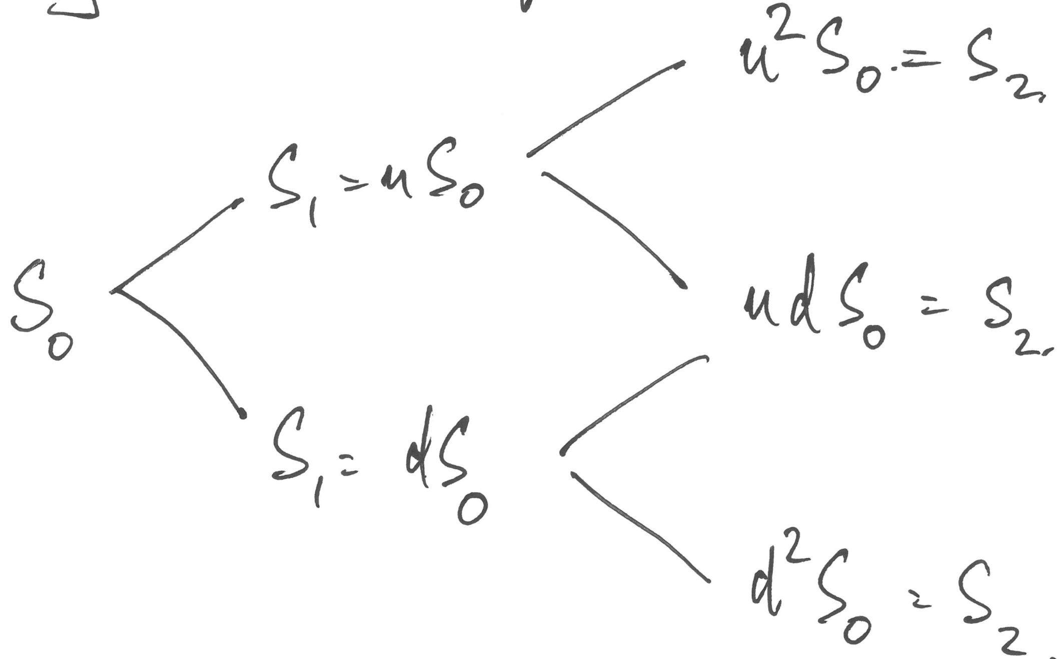
4. **Multi-Period Binomial Model.**

- Same setup as the one period case $0 < d < 1 + r < u$, and toss coins that land heads with probability p_1 and tails with probability q_1.
- Except now the security matures at time $N > 1$.
- Stock price: $S_{n+1} = uS_n$ if $n + 1$-th coin toss is heads, and $S_{n+1} = dS_n$ otherwise.
- To replicate it a security, we start with capital X_0.
- Buy Δ_0 shares of stock, and put the rest in cash.
- Get $X_1 = \Delta_0 S_1 + (1 + r)(X_0 - \Delta_0 S_0)$.
- Repeat. *Self Financing Condition:* $X_{n+1} = \Delta_n S_{n+1} + (1 + r)(X_n - \Delta_n S_n)$.
- *Adaptedness:* Δ_n can only depend on outcomes of coin tosses before n!

Coin flip → Heads → multiply stock price by u.

→ Tails → " " " " d.

Security matures after N time periods $(N > 1$.



Wealth evolution:

$X_0 \rightarrow$ initial wealth.

$\Delta_0 \rightarrow$ # shares of stock bought at time 0.

$X_1 \rightarrow$ wealth at time 1: $X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0)$.

At time 1: Change pos. $\rightarrow$ hold Δ_1 shares of stock.

$$X_2 = \Delta_1 S_2 + (1+r)(X_1 - \Delta_1 S_1)$$

$\vdots$

$$\boxed{X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n).}$$

Self financing. (No external cash flow)

<u>Adaptedness</u>: Δ_n can only use outcomes of coin tosses before (or at) time n.

$\rightarrow$ NOT outcomes of coin tosses after <u>time n</u>.

$\boxed{\textbf{Proposition 4.1.}}$ *Consider a security that pays V_N at time N. Then for any $n \leqslant N$:*

$$V_n = \frac{1}{1+r} \tilde{E}_n V_N \,, \qquad \Delta_n = \frac{V_{n+1}(\omega_{n+1} = H) - V_{n+1}(\omega_{n+1} = T)}{(u-d) S_n} \,.$$

- *V_n is the arbitrage free price at time $n \leqslant N$.*
- *Δ_n is the number of shares held in the replicating portfolio at time n (trading strategy).*

Question 4.2. *Why does this work?*

Question 4.3. *What is $\tilde{\mathbf{E}}_n$? (It's different from $\mathbf{E}$, and different from $\mathbf{E}_n$).*

A security with payoff V_N can be replicated.

~~APF~~ AFP at time $n \leq N$

$= $ Wealth of Rep port at time n

$$= \frac{1}{(1+r)^{N-n}} \cdot \tilde{E}_n V_N \,.$$

IOU.

4.1. **Quick review probability (finite Sample spaces).** This is just a quick reminder, to fix notation. Read one of the references, or look over the prep material / videos for a more through treatment. The only thing we will cover in any detail is conditional expectation.

Let $N \in \mathbb{N}$ be large (typically the maturity time of financial securities).

Definition 4.4. The *sample space* is the set $\Omega = \{(\omega_1, \dots, \omega_N) \mid \text{each } \omega_i \text{ represents the outcome of a coin toss.}\}$

▷ E.g. $\omega_i \in \{H, T\}$, or $\omega_i \in \{\pm 1\}$. (Each ω_i could also represent the outcome of the roll of a M sided die.)

Definition 4.5. A *sample point* is a point $\omega = (\omega_1, \dots, \omega_N) \in \Omega$.

▷ Each sample point represents the outcome of a sequence of *all* coin tosses from 1 to N.

Definition 4.6. A *probability mass function* is a function $p \colon \Omega \to [0,1]$ such that $\sum_{\omega \in \Omega} p(\omega) = 1$.

Example 4.7. Typical example: Fix $p_1 \in (0,1)$, $q_1 = 1 - p_1$ and set $p(\omega) = p_1^{H(\omega)} q_1^{T(\omega)}$. Here $H(\omega)$ is the number of heads in the sequence $\omega = (\omega_1, \dots, \omega_N)$, and $T(\omega)$ is the number of tails.

Definition 4.8. An event is a subset of Ω. Define $\boldsymbol{P}(A) = \sum_{\omega \in A} p(\omega)$.

Example 4.9. $A\{\omega \in \Omega \mid \omega_1 = +1\}$. Check $\boldsymbol{P}(A) = p_1$.

+1 = Heds, −1 = Tails.

outcome of a cointoss.

$\Omega \to$ sample space

$= \{(\omega_1, \omega_2, \dots \omega_N) \mid \omega_i \in \{\pm 1\}\}$

$\omega = (\omega_1, \dots \omega_N) \in \Omega \leftarrow$ sample point.

PMF: $p : \Omega \to [0,1]$. & $\sum_{\omega \in \Omega} p(\omega) = 1$.

$p(\omega) \approx$ prob that this ~~x~~ particular seq of ~~ω~~ occurs.

$\underline{\underline{A}} \subseteq \Omega$ (any subset) $\rightarrow$ called an event.

~~Ex~~ ~~Ex~~ Eg: fix $p_1 \in (0,1)$, $q_1 = 1 - p_1$

$$p(\omega) = p_1^{H(\omega)} \cancel{p_2} q_1^{T(\omega)}$$

$H(\omega) =$ # heads in the seq $(\omega_1, \cdots \omega_N)$.

$T(\omega) =$ " tails " " " "

$$P(A) = \text{prob that the event } A \text{ occurs} = \sum_{\omega \in A} p(\omega).$$

4.2. **Random Variables and Independence.**

Definition 4.10. A *random variable* is a function $X : \Omega \to \mathbb{R}$.

Example 4.11. $X(\omega) = \begin{cases} 1 & \omega_2 = +1\,, \\ -1 & \omega_2 = -1\,, \end{cases}$ is a random variable corresponding to the outcome of the second coin toss.

A Random Var. is fn $X : \Omega \longrightarrow \mathbb{R}$.

Definition 4.12. The *expectation* of a random variable X is $\boldsymbol{E}X = \sum X(\omega)p(\omega)$.

Remark 4.13. Note if $\text{Range}(X) = \{x_1, \ldots, x_n\}$, then $\boldsymbol{E}X = \sum X(\omega)p(\omega) = \sum_1^n x_i \boldsymbol{P}(X = x_i)$.

Definition 4.14. The *variance* of a random variable is $\text{Var}(X) = \boldsymbol{E}(X - \boldsymbol{E}X)^2$.

Remark 4.15. Note $\text{Var}(X) = \boldsymbol{E}X^2 - (\boldsymbol{E}X)^2$.

$$\mathbb{E}X = \text{"mean"} = \text{"avge of } X\text{"}.$$

$$= \sum X(\omega)\, p(\omega)$$

$$= \sum_{x_i \in \text{Range}(X)} x_i \, P(X = x_i).$$

Definition 4.16. Two events are independent if $\boldsymbol{P}(A \cap B) = \boldsymbol{P}(A)\boldsymbol{P}(B)$.

Definition 4.17. The events $A_1, \ldots, A_n$ are independent if for any sub-collection $A_{i_1}, \ldots, A_{i_k}$ we have

$$\boldsymbol{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \boldsymbol{P}(A_{i_1})\boldsymbol{P}(A_{i_2}) \cdots \boldsymbol{P}(A_{i_k}).$$

Remark 4.18. When $n > 2$, it is not enough to only require $\boldsymbol{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \boldsymbol{P}(A_1)\boldsymbol{P}(A_2) \cdots \boldsymbol{P}(A_n)$

Definition 4.19. Two random variables are independent if $\boldsymbol{P}(X = x,\ Y = y) = \boldsymbol{P}(X = x)\boldsymbol{P}(Y = y)$ for all $x, y \in \mathbb{R}$.

Definition 4.20. The random variables $X_1,\ \dots,\ X_n$ are independent if for all $x_1, \dots, x_n \in \mathbb{R}$ we have

$$\boldsymbol{P}(X_1 = x_1,\ X_2 = x_2, \dots, X_n = x_n) = \boldsymbol{P}(X_1 = x_1)\boldsymbol{P}(X_2 = x_2) \cdots \boldsymbol{P}(X_n = x_n)\,.$$

Remark 4.21. Independent random variables are uncorrelated, but not vice versa.

4.3. **Filtrations.**

Definition 4.22. We define a *filtration* on Ω as follows:

▷ $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

▷ $\mathcal{F}_1$ = all events that can be described by only the first coin toss. E.g. $A = \{\omega \mid \omega_1 = +1\} \in \mathcal{F}_1$.

▷ $\mathcal{F}_n$ = all events that can be described by only the first n coin tosses. E.g. $A = \{\omega \mid \omega_1 = 1, \omega_3 = -1, \omega_n = 1\} \in \mathcal{F}_n$.

Remark 4.23. Note $\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_N = \mathcal{P}(\Omega)$.

Remark 4.24. If $A, B \in \mathcal{F}_n$, then so do A^c, B^c, $A \cap B$, $A \cup B$, $A - B$, $B - A$.

$\mathcal{F}_0 = \{\emptyset, \Omega\} \leftarrow$ "info you have before tossig any coins".

$\mathcal{F}_1$ = All events that can be described using ONLY the first coin toss. (E.g. $A = \{1^{st}$ toss is heads$\} \in \mathcal{F}_1$

$B = \{2^{nd}$ toss is tails$\} \notin \mathcal{F}_1$.

$\vdots$

$\mathcal{F}_n$ = All events that can be desc using only the first n coin tosses.

Definition 4.25. Let $n \in \{0, \ldots, N\}$. We say a random variable X is $\mathcal{F}_n$-measurable if $X(\omega)$ only depends on $\omega_1, \ldots, \omega_n$.
▷ Equivalently, for any $B \subseteq \mathbb{R}$, the event $\{X \in B\} \in \mathcal{F}_n$.

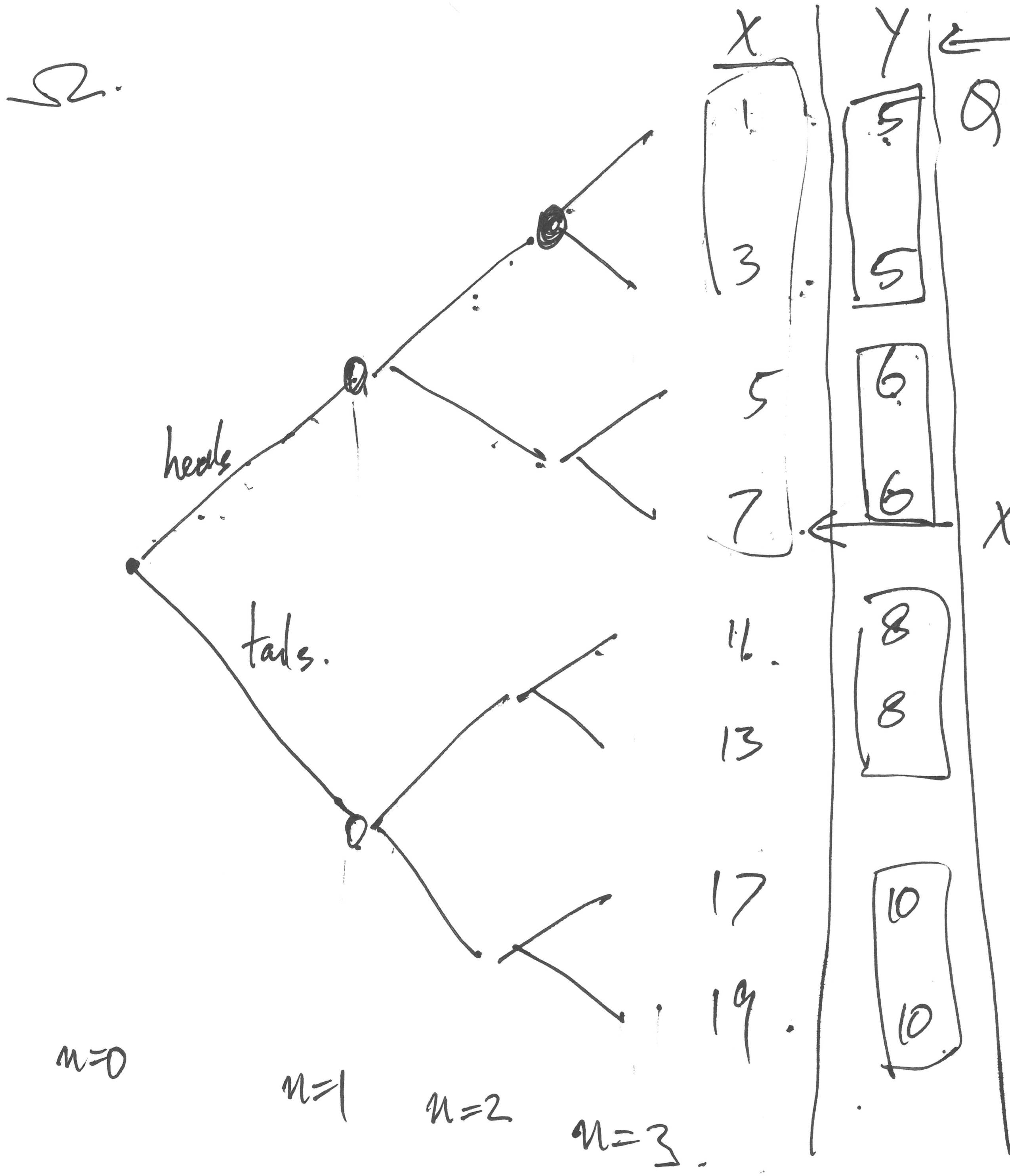
Remark 4.26 (Use in Finance). For every n, the trading strategy at time n (denoted by Δ_n) must be $\mathcal{F}_n$ measurable. We can not trade today based on tomorrows price.

Example 4.27. If we represent Ω as a tree, $\mathcal{F}_n$ measurablity can be visualized by checking constancy on leaves.

① X is $\mathcal{F}_n$ meas if $X(\omega)$ only depends on $\omega_1, \omega_2 \cdots \omega_n$ & not $\omega_{n+1} \cdots \omega_N$.

② X is $\mathcal{F}_n$-meas. $\Longleftrightarrow$ For any $B \subseteq \mathbb{R}$, $\{X \in B\} \in \mathcal{F}_n$.

③ Finance: $\Delta_n \rightarrow$ always has to be $\mathcal{F}_n$-meas.



Y is $\mathcal{F}_2$ meas.

Q: Is X $\mathcal{F}_2$ - meas.

(No).

X = 7 if toss H, T, T

4.4. **Conditional expectation.**

Definition 4.28. Let X be a random variable, and $n \leqslant N$. We define $\boldsymbol{E}(X \mid \mathcal{F}_n) = \boldsymbol{E}_n X$ to be the *random variable* given by

$$\boldsymbol{E}_n X(\omega) = \sum_{x_i \in \text{Range}(X)} x_i \boldsymbol{P}(X = x_i \mid \Pi_n(\omega)), \qquad \text{where} \qquad \Pi_n(\omega) = \{\omega' \in \Omega \mid \omega_1' = \omega_1, \ \ldots, \omega_n' = \omega_n\}$$

Remark 4.29. $\boldsymbol{E}_n X$ is the "best approximation" of X given only the first n coin tosses.

Remark 4.30. The above formula does not generalize well to infinite probability spaces. We will develop certain properties of $\boldsymbol{E}_n$, and then only use those properties going forward.

Example 4.31. If we represent Ω as a tree, $\boldsymbol{E}_n X$ can be computed by averaging over leaves.

$E_n X(\omega) =$ cond exp of X given $\mathcal{F}_n$.

$= E(X \mid \mathcal{F}_n)$

$= \sum_{x_i \in \text{Range}(X)} x_i P(X = x_i \mid \Pi_n(\omega))$.

$\underline{\Pi_n(\omega)} = \{ \omega' \mid \omega_1 = \omega_1', \ \omega_2 = \omega_2' \cdots \omega_n = \omega_n' \}$.

Last time: $\boxed{\text{No Arb}}$ (assumption)

Bond

Stocks

Market

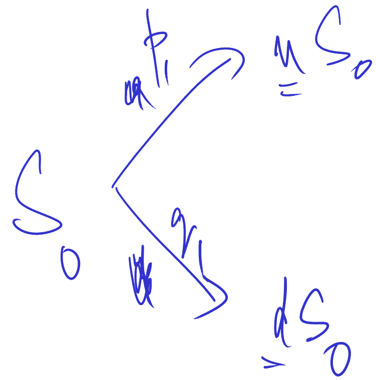
NTA (option)

AFP: V_0 → if when allowed to trade the NTA at price V_0 the extended market remains arb free.

Replication!

If the payoff of the NTA can be replicated using only tradable assets then X_{0n} = initial wealth of the rep port $= V_{0n} =$ AFP.

Last time: Binom model: Multi period version



Formula: Payoff = V_N

$$\boxed{AFP = \frac{1}{D_n}\tilde{E}_n(D_N V_N)}$$

4.4. Conditional expectation.

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$$\boldsymbol{E}_n X(\omega) = \sum_{x_i \in \mathrm{Range}(X)} x_i \boldsymbol{P}(X = x_i \mid \Pi_n(\omega))$$

where $$\Pi_n(\omega) = \{\omega' \in \Omega \mid \omega'_1 = \omega_1, \ldots, \omega'_n = \omega_n\}$$

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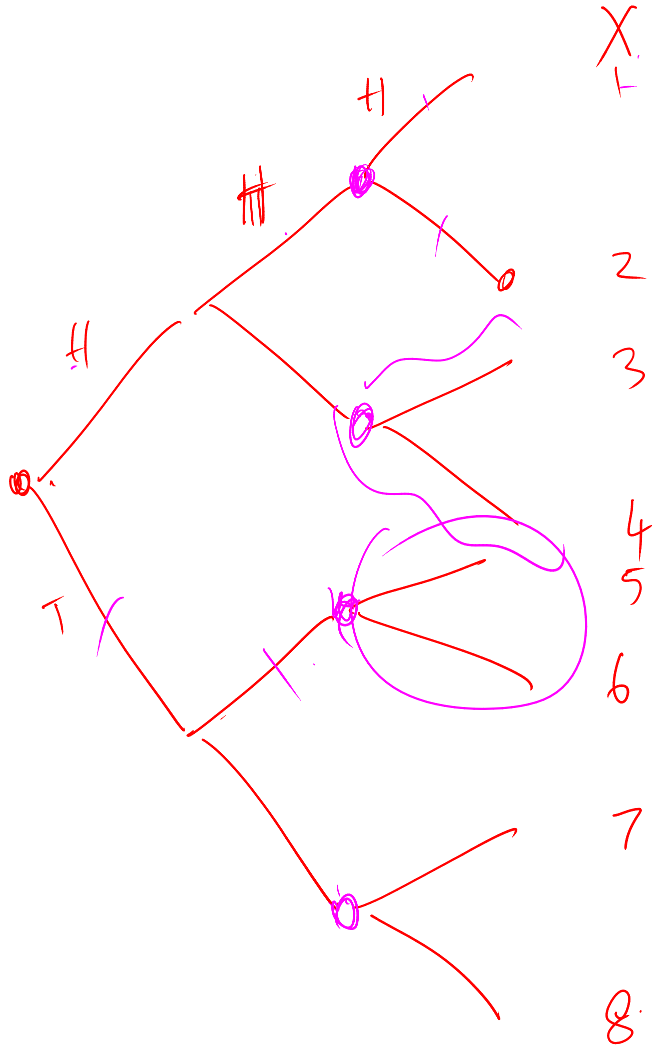
Example 4.30. If we represent Ω as a tree, $\boldsymbol{E}_n X$ can be computed by averaging over leaves.

Remark 4.31. $\boldsymbol{E}_n X$ is the "best approximation" of X given only the first n coin tosses.

$E_n X \longrightarrow$ "Best approximation of X given info up to time n"



$E_2 X$

$$p_1 \cdot 1 + q_1 \cdot 2$$

$$p_1 \cdot 3 + q_1 \cdot 4$$

$$p_1 \, 3 + q_1 \cdot 4$$

(3 coin tosses)

Proposition 4.32. *The conditional expectation $\boldsymbol{E}_n X$ defined by the above formula satisfies the following two properties:*

(1) $\boldsymbol{E}_n X$ *is an* $\mathcal{F}_n$*-measurable* random variable.

(2) *For every* $A \in \mathcal{F}_n$, $\displaystyle\sum_{\omega \in A} \boldsymbol{E}_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$.

($E_n X$ dep only on first n coin tosses.)

Remark 4.33. This property is used to define conditional expectations in the continuous time setting. It turns out that there is exactly one random variable that satisfies both the above properties; and thus we define $\boldsymbol{E}_n X$ to be the unique random variable which satisfies both the above properties.

Remark 4.34. Note, choosing $A = \Omega$, we see $\boxed{\boldsymbol{E}(\boldsymbol{E}_n X) = \boldsymbol{E} X}$.

On A: Avg of $X = \dfrac{1}{\cancel{P(A)}} \sum_{\omega \in A} X(\omega) p(\omega)$

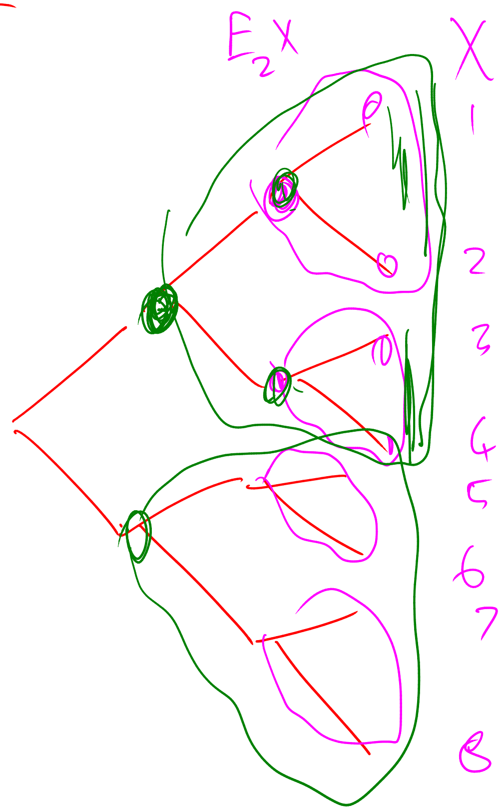
Avg of $E_n X$ (on A) $= \dfrac{1}{\cancel{P(A)}} \sum_{\omega \in A} E_n X(\omega) p(\omega)$

equal.

Proposition 4.35. *(1) If X, Y are two random variables and $\alpha \in \mathbb{R}$, then $\boldsymbol{E}_n(X + \alpha Y) = \boldsymbol{E}_n X + \alpha \boldsymbol{E}_n Y$.*

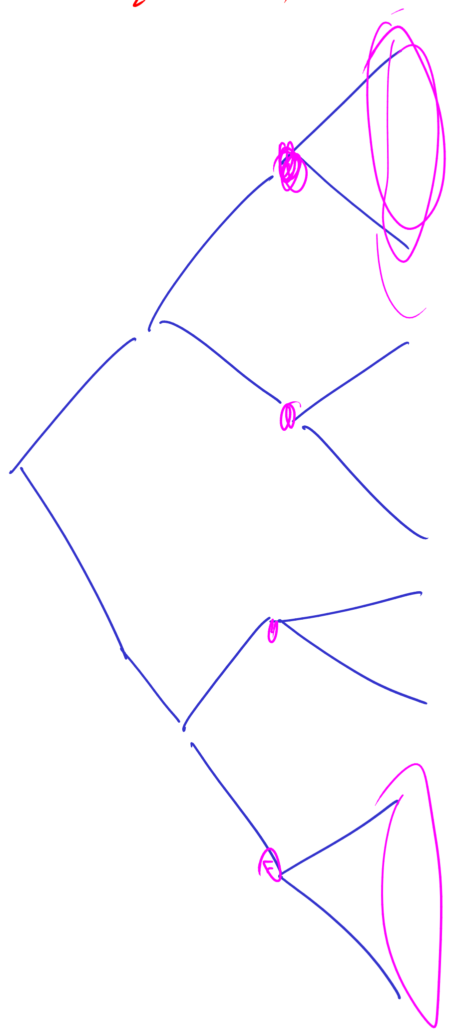
(2) (Tower property) *If $m \leqslant n$, then $\boldsymbol{E}_m(\boldsymbol{E}_n X) = \boldsymbol{E}_m X$.*

(3) If X is $\mathcal{F}_n$ measurable, and Y is any random variable, then $\boldsymbol{E}_n(XY) = X\boldsymbol{E}_n Y$.



$E_2 X$ $\quad$ $E_1 X$

(X is $\mathcal{F}_2$ meas) $\longrightarrow$



X	Y	$E_2(XY)$
100	1	$100(1p_1 + 2q_1)$
100	2	
300	3	
300	4	
-10	5	
-10	6	
5	7	
5	8	

Proposition 4.36. *(1) If X is measurable with respect to $\mathcal{F}_n$, then $\boldsymbol{E}_n X = X$.*
(2) If X is independent of $\mathcal{F}_n$ then $\boxed{\boldsymbol{E}_n X = \boldsymbol{E} X}$.

Remark 4.37. We say X is independent of $\mathcal{F}_n$ if for every $A \in \mathcal{F}_n$ and $B \subseteq \mathbb{R}$, the events A and $\{X \in B\}$ are independent.

Example 4.38. If X only depends on the $(n+1)^{\text{th}}$, $(n+2)^{\text{th}}$, $\dots$, N^{th} coin tosses and *not* the 1^{st}, 2^{nd}, $\dots$, n^{th} coin tosses, then X is independent of $\mathcal{F}_n$.

Notation: $\{X \in B\} = \{\omega \mid X(\omega) \in B\}$

Proposition 4.39 (Independence lemma)**.** *If X is independent of $\mathcal{F}_n$ and Y is $\mathcal{F}_n$-measurable, and $f : \mathbb{R}^2 \to \mathbb{R}$ is a function then*

$$\boldsymbol{E}_n f(X, Y) = \sum_{i=1}^{m} f(x_i, Y) \boldsymbol{P}(X = x_i)\,, \qquad \textit{where } \{x_1, \dots, x_m\} = X(\Omega)\,.$$

Range(X)

Say $Y = y$ (some const)

$$E_n f(X, y) = \boxed{E f(X, y)} = \sum_{x_i \in \text{Range}(X)} f(x_i, y)\, P(X = x_i)$$

Y =

Y $\qquad$ Y

Avrage the ind RV treating the meas RV as a const.

4.5. **Martingales.**

Definition 4.40. A *stochastic process* is a collection of random variables $X_0, X_1, \ldots, X_N$.

Example 4.41. Typically X_n is the wealth of an investor at time n, or S_n is the price of a stock at time n.

Definition 4.42. A stochastic process is $\boxed{\textit{adapted}}$ if X_n is $\mathcal{F}_n$-measurable for all n. (Non-anticipating.)

Remark 4.43. Requiring processes to be adapted is fundamental to Finance. Intuitively, being adapted forbids you from trading today based on tomorrows stock price. All processes we consider (prices, wealth, trading strategies) will be adapted.

Example 4.44 (Money market). Let $Y_0 = Y_0(\omega) = a \in \mathbb{R}$. Define $Y_{n+1} = (1+r)Y_n$. (Here r is the interest rate.)

Example 4.45 (Stock price). Let $S_0 \in \mathbb{R}$. Define $S_{n+1}(\omega) = \begin{cases} uS_n(\omega) & \omega_{n+1} = 1\,, \\ dS_n(\omega) & \omega_{n+1} = -1\,. \end{cases}$

S_n is a stoch process

S_n is an ADAPTED process

$Y_n = (1+r)^n a.$

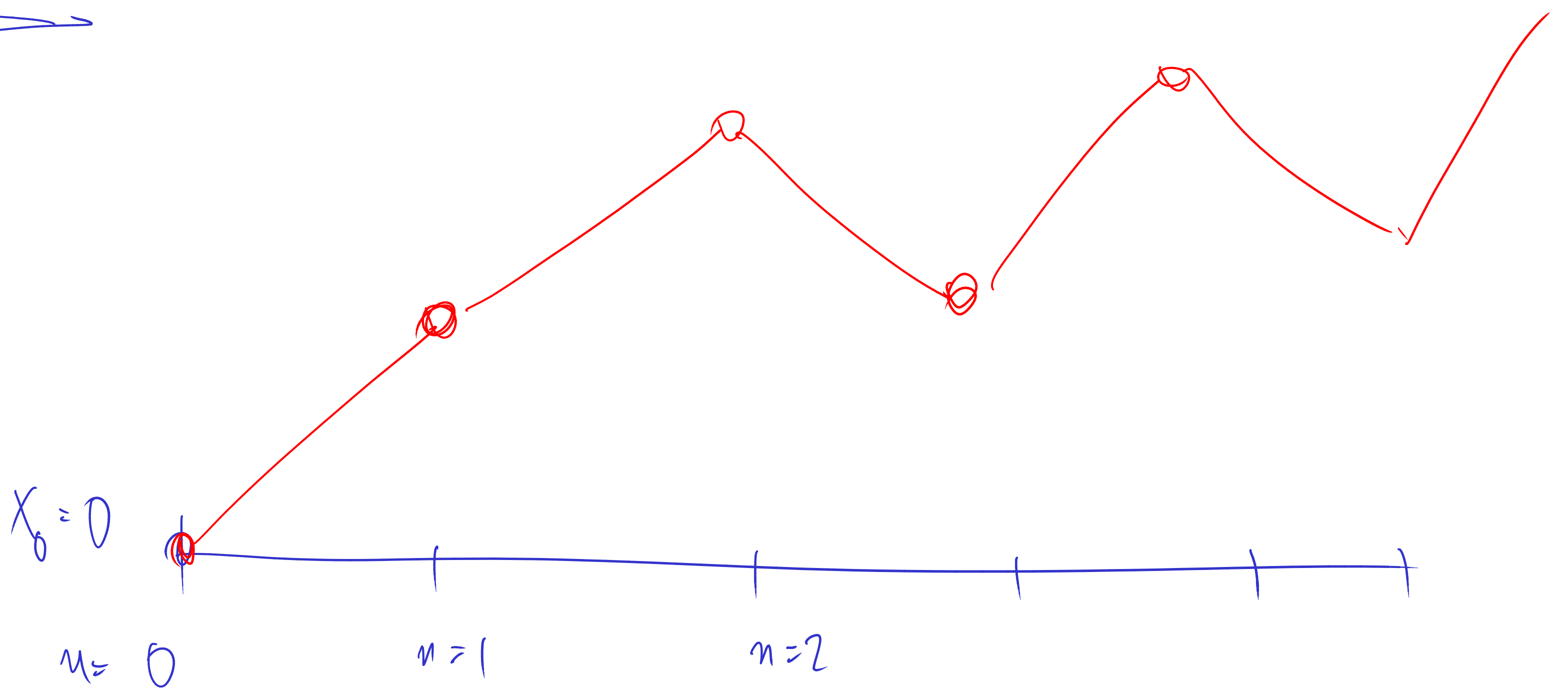
(adapted process)

Definition 4.46. We say an adapted process M_n is a martingale if $\boldsymbol{E}_n M_{n+1} = M_n$. (Recall $\boldsymbol{E}_n Y = \boldsymbol{E}(Y \mid \mathcal{F}_n)$.)

Remark 4.47. Intuition: A martingale is a "fair game".

$E\xi_n = 0 \ \forall n.$

Example 4.48 (Unbiased random walk). If $\xi_1, \dots, \xi_N$ are i.i.d. and mean zero, then $X_n = \sum_{k=1}^n \xi_k$ is a martingale.



Check X is a mg: Note $X_{n+1} = X_n + \xi_{n+1}$

$$\Rightarrow E_n(X_{n+1}) = E_n(X_n + \xi_{n+1})$$

$$= \underbrace{E_n X_n}_{X_n \text{ is } \mathcal{F}_n \text{ meas}} + \underbrace{E_n \xi_{n+1}}_{\xi_{n+1} \text{ is ind of } \mathcal{F}_n}$$

$$= X_n + E_{\cancel{n}} \xi_{n+1} \xrightarrow{} 0$$

$$\Rightarrow E_n X_{n+1} = X_n$$

$\Rightarrow X$ is a mg!

Remark 4.49. If M is a martingale, then for every $m \leqslant n$, we must have $\boldsymbol{E}_m M_n = M_m$.

Remark 4.50. If M is a martingale then $\boldsymbol{E} M_n = \boldsymbol{E} M_0 = M_0$.

$$E_n M_{n+2} \overset{\text{tower}}{=} E_n E_{n+1} M_{n+2} \overset{\text{Mg}}{=} E_n M_{n+1} \overset{\text{Mg}}{=} M_n$$

Note

$$E M_n = E_0 M_n \overset{4.49}{=} M_0$$

4.6. **Change of measure.**

- Gambling in a Casino: If it's a martingale, then on average you won't make or lose money.
- Stock market: Bank always pays interest! Not looking for a "break even" strategy.
- Mathematical tool that helps us price securities: Find a *Risk Neutral Measure.*
  - ▷ Discounted stock price is (usually) not a martingale.
  - ▷ Invent a "risk neutral measure" which the discounted stock price is a martingale.
  - ▷ Securities can be priced by taking a conditional expectation *with respect to the risk neutral measure*. (That's the meaning of $\tilde{\boldsymbol{E}}_n$ in Proposition 4.1.)

AFP at time $n = \frac{1}{D_n} \tilde{E}_n (D_N V_D)$

Cond exp wrt the "Risk Neutral Measure"

Definition 4.51. Let $D_n = (1+r)^{-n}$ be the discount factor. (So D_n\$ in the bank at time 0 becomes 1\$ in the bank at time n.)

- Invent a new probability mass function $\tilde{p}$.
- Use a tilde to distinguish between the new, invented, probability measure and the old one.
  - ▷ $\tilde{\boldsymbol{P}}$ the probability measure obtained from the PMF $\tilde{p}$ (i.e. $\tilde{\boldsymbol{P}}(A) = \sum_{\omega \in A} \tilde{p}(\omega)$).
  - ▷ $\tilde{\boldsymbol{E}}$, $\tilde{\boldsymbol{E}}_n$ conditional expectation with respect to $\tilde{\boldsymbol{P}}$ (the new "risk neutral" coin)

Definition 4.52. We say $\boldsymbol{P}$ and $\tilde{\boldsymbol{P}}$ are equivalent if for every $A \in \mathcal{F}_N$, $\boldsymbol{P}(A) = 0$ if and only if $\tilde{\boldsymbol{P}}(A) = 0$.

Definition 4.53. A *risk neutral measure* is an equivalent measure $\tilde{\boldsymbol{P}}$ under which $D_n S_n$ is a martingale. (I.e $\tilde{\boldsymbol{E}}_n(D_{n+1}S_{n+1}) = D_n S_n$. for all n)

Remark 4.54. If there are more than one risky assets, S^1, …, S^k, then we require $D_n S_n^1$, …, $D_n S_n^k$ to all be martingales under the risk neutral measure $\tilde{\boldsymbol{P}}$.

Remark 4.55. Proposition 4.1 says that any security with payoff V_N at time N has arbitrage free price $V_n = \frac{1}{D_n}\tilde{\boldsymbol{E}}_n(D_N V_N)$ at time n. (Called the risk neutral pricing formula.)

tilde

Proposition 4.56. *Let $\tilde{\boldsymbol{P}}$ be an equivalent measure under which the coins are i.i.d. and land heads with probability $\tilde{p}_1$ and tails with probability $\tilde{q}_1 = 1 - \tilde{p}_1$.*

(1) *Under $\tilde{\boldsymbol{P}}$, we have $\tilde{\boldsymbol{E}}_n(D_{n+1}S_{n+1}) = \frac{\tilde{p}_1 u + \tilde{q}_1 d}{1+r} D_n S_n$.*

(2) *$\tilde{\boldsymbol{P}}$ is the risk neutral measure if and only if $\tilde{p}_1 u + \tilde{q}_1 d = 1 + r$. (Explicitly $\tilde{p}_1 = \frac{1+r-d}{u-d}$, and $\tilde{q}_1 = \frac{u-(1+r)}{u-d}$.)*

(1) $\tilde{E}_n(D_{n+1} S_{n+1}) = (1+r)^{-(n+1)} \tilde{E}_n S_{n+1}$

$= (1+r)^{-(n+1)} \tilde{E}_n \left(S_n X_{n+1} \right)$

where $X_{n+1} = \begin{cases} u & \omega_{n+1} = 1 \\ d & \omega_{n+1} = -1 \end{cases}$

$= (1+r)^{-(n+1)} \tilde{E}_n \left(S_n X_{n+1} \right)$

A

(S_n is $\mathcal{F}_n$ meas, X_{n+1} is ind)

$$= (1+r)^{-(n+1)} S_n \tilde{E} X_{n+1}$$

$$= (1+r)^{-n+1} S_n \left(\tilde{p}_1 u + \tilde{r}_1 d \right)$$

$$= \left(\frac{\tilde{p}_1 u + \tilde{q}_1 d}{1+r} \right) D_n S_n$$

//

Theorem 4.57. *Let X_n represent the wealth of a portfolio at time n. The portfolio is self-financing portfolio if and only if the discounted wealth $D_n X_n$ is a martingale under the risk neutral measure $\tilde{\boldsymbol{P}}$.*

Remark 4.58. Recall a portfolio is *self financing* if $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$ for some *adapted* process Δ_n.

(1) That is, self-financing portfolios use only tradable assets when trading, and don't look into the future.

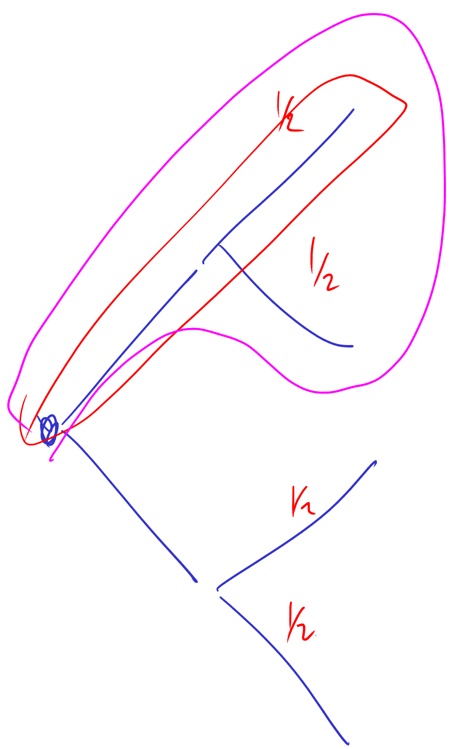
(2) All replication has to be done using self-financing portfolios.

Check: Self fin $\Rightarrow$ $D_n X_n$ is a $\tilde{P}$ mg.

$$\tilde{E}_n\left(D_{n+1} X_{n+1}\right) = \tilde{E}_n\left(D_{n+1}\left(\Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)\right)\right)$$

$$= \Delta_n \underbrace{\tilde{E}_n\left(D_{n+1} S_{n+1}\right)}_{D_n S_n} + D_n\left(X_n - \Delta_n S_n\right)$$

$$= D_n X_n \qquad \text{QED.}$$



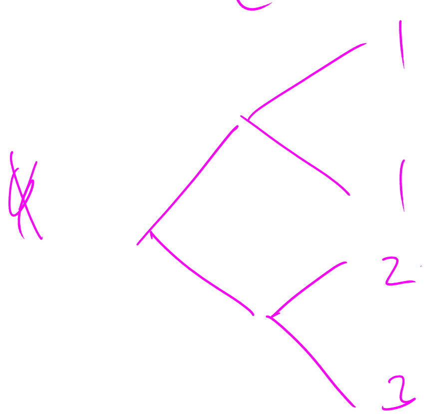
X	$F_Y\,X$
1	3/2
2	3/2
3	7/2
4	7/2

$A = \{(1,1)\} \in \mathcal{F}_2$

$\notin \mathcal{F}_1$

$B = \{(1,1),(1,-1)\} \in \mathcal{F}_1$

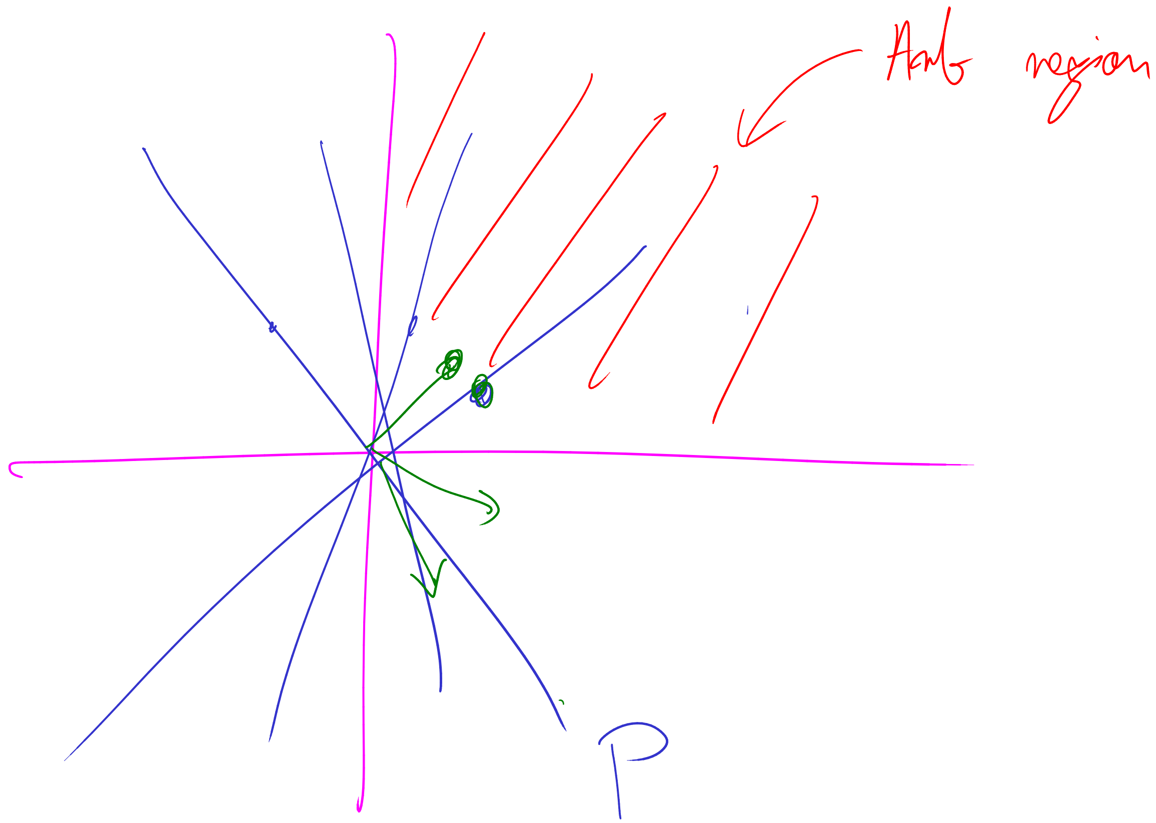
$$\sigma(X) = \{\{X \in B\} \mid B \subseteq \mathbb{R}\}$$



$\sigma(X) = \{\{(1,1),(1,-1)\},$

$\{(-1,1)\}$ $\{(-1,-1)\}$

$\{(-1,1),(-1,-1)\}, \emptyset, \Omega$


Amb region
P

Last time:

$$\tilde{P} \longrightarrow \tilde{E}_n\left(D_{n+1} S_{n+1}\right) = D_n S_n$$

(Discounted stock is a Mg under $\tilde{P}$)

Risk neutral Measure.

Theorem 4.57. *Let X_n represent the wealth of a portfolio at time n. The portfolio is self-financing portfolio if and only if the discounted wealth $D_n X_n$ is a martingale under the risk neutral measure $\tilde{\boldsymbol{P}}$.*

Remark 4.58. Recall a portfolio is *self financing* if $\boxed{X_{n+1} = \boxed{\Delta_n} S_{n+1} + (1+r)(X_n - \Delta_n S_n)}$ for some *adapted* process Δ_n.

(1) That is, self-financing portfolios use only tradable assets when trading, and don't look into the future.
(2) All replication has to be done using self-financing portfolios.

Last time: $X_n \longrightarrow$ wealth of a self fin Port

$\Rightarrow D_n X_n$ is a mg under $\tilde{P}$.

Conversely: $D_n X_n$ mg under $\tilde{P} \Rightarrow X_n$ is self fin $\longrightarrow$ You try (No time)

Proof of Proposition 4.1. → Security pays V_N at time N

Then AFP at time $n \leq N$ is

$$V_n = \frac{1}{D_n} \tilde{E}_n (D_N V_N).$$

Pf: Price by replication.

Find a self fin port ~~&~~ $\Big\}$ Want $X_N = V_N$.

wealth at time $\cancel{X}n \to X_n$

Then we know X_n = AFP.

① Choose $X_N = V_N$

② Define $X_n = \underbrace{\frac{1}{D_n} \tilde{E}_n \left(D_N X_N \right)} = \frac{1}{D_n} \tilde{E}_n \left(D_N V_N \right)$

③ Claim: $D_n X_n$ is a Mg under $\tilde{P}$

Pf: $\tilde{E}_n \left(D_{n+1} X_{n+1} \right)$ Want $D_n X_n$.

Know $\tilde{E}_n(D_{n+1} X_{n+1}) = \tilde{E}_n\left(\tilde{E}_{n+1}(D_N X_N)\right)$

$$\overset{\text{tower}}{=} \tilde{E}_n(D_N X_N) = \underline{D_n X_n}$$

Done !!

(4) Thm 4.57 $\Rightarrow$ X_n = wealth of a self fin Port.

Know $X_N = V_N$ $\Rightarrow$ Replication $\Rightarrow$ $\forall n \leq N$, X_n = AFP.

Done !!

Example 4.59. Consider two stocks S^1 and S^2, $u = 2$, $d = 1/2$.

- The coin flips for S^1 are heads with probability 90%, and tails with probability 10%.
- The coin flips for S^2 are heads with probability 99%, and tails with probability 1%.
- Which stock do you like more?
- Amongst a call option for the two stocks with strike K and maturity N, which one will be priced higher?

Same!

Formula for $\tilde{p} = \dfrac{1+r-d}{u-d}$ ← doesn't depend on p, q!!

Remark 4.60. Even though the stock price changes according to a coin that flips heads with probability p_1, the arbitrage free price is computed using conditional expectations using the *risk neutral probability*. So when computing $\tilde{\boldsymbol{E}}_n V_N$, we use our new invented "risk neutral" coin that flips heads with probability $\tilde{p}_1$ and tails with probability $\tilde{q}_1$.

Concepts that will be generalized to continuous time.

- Probability measure: Lebesgue integral, and not a finite sum. Same properties.
- Filtration: Same intuition. No easy description.
- Conditional expectation: Same properties, no formula.
- Risk neutral measure: Formula for $\tilde{\boldsymbol{P}}$ is complicated (Girsanov theorem.)
- Everything still works because of of Theorem 4.57. Understanding why is harder.

Better way: $E\xi_i^2 = 1$

not essential.

5. Stochastic Processes

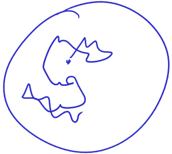
5.1. Brownian motion.

- Discrete time: Simple Random Walk.
  - ▷ $X_n = \sum_1^n \xi_i$, where ξ_i's are i.i.d. $\boldsymbol{E}\xi_i = 0$, and Range$(\xi_i) = \{\pm 1\}$.
- Continuous time: Brownian motion.
  - ▷ $Y_t = X_n + (t - n)\xi_{n+1}$ if $t \in [n, n+1)$.
  - ▷ Rescale: $Y_t^\varepsilon = \sqrt{\varepsilon}\, Y_{t/\varepsilon}$. (Chose $\sqrt{\varepsilon}$ factor to ensure $\operatorname{Var}(Y_t^\varepsilon) \approx t$.)
  - ▷ Let $W_t = \lim_{\varepsilon \to 0} Y_t^\varepsilon$.

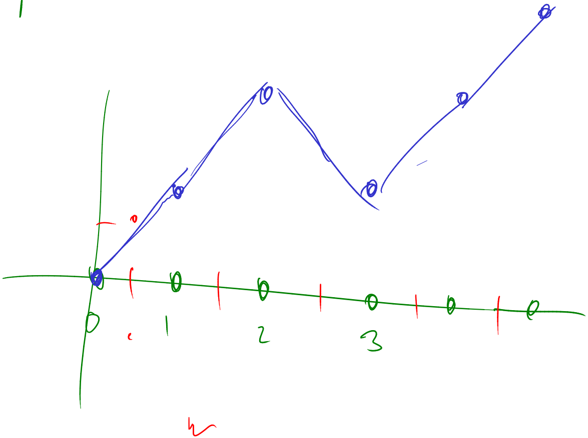
Definition 5.1 (Brownian motion)**.** The process W above is called a Brownian motion.

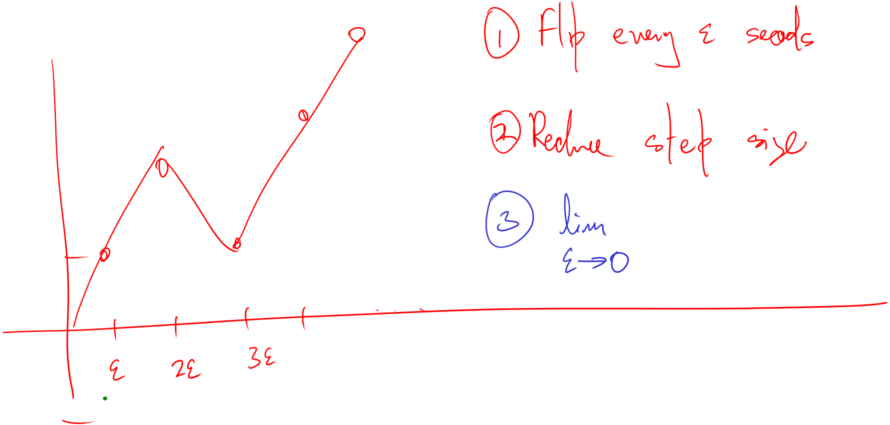
▷ Named after Robert Brown (a botanist).

▷ Definition is intuitive, but not as convenient to work with.



Simple random Walk





① Flip every ε seconds

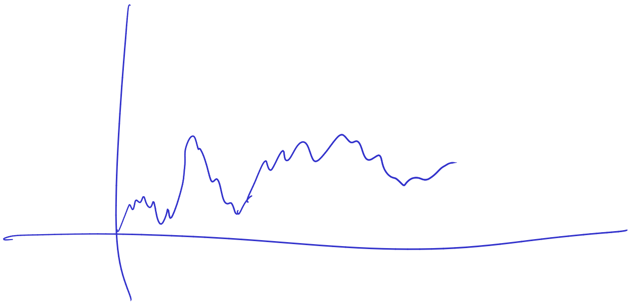
② Reduce step size

③ $\lim_{\varepsilon \to 0}$

$\underline{\varepsilon \to 0}$ converge to a "cts time RW"

(Brownian motion)

u



$Y_{t/\varepsilon}$ (say $t/\varepsilon \in \mathbb{N}$)

$$Y_{t/\varepsilon} = \sum_{1}^{t/\varepsilon} \xi_i$$ (sum of $\frac{t}{\varepsilon}$ iid RV's mean 0 & Var 1)

$$\mathrm{Var}(Y_{t/\varepsilon}) = \frac{t}{\varepsilon}$$

$$\Rightarrow \mathrm{Var}(\sqrt{\varepsilon}\, Y_{t/\varepsilon}) = (\sqrt{\varepsilon})^2 \cdot \frac{t}{\varepsilon} = t$$

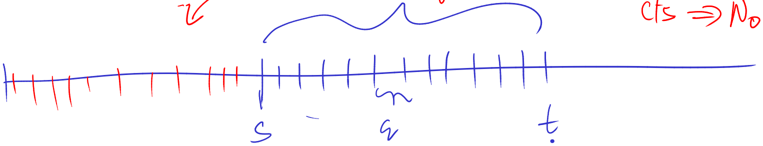
- If t, s are multiples of ε: $Y_t^\varepsilon - Y_s^\varepsilon \sim \sqrt{\varepsilon} \sum_{i=1}^{(t-s)/\varepsilon} \xi_i \xrightarrow{\varepsilon \to 0} \mathcal{N}(0, t-s)$. (CLT)
- $Y_t^\varepsilon - Y_s^\varepsilon$ only uses coin tosses that are "after s", and so independent of Y_s^ε.

Definition 5.2. Brownian motion is a *continuous process* such that:

(1) $W_t - W_s \sim \mathcal{N}(0, t-s)$,
(2) $W_t - W_s$ is independent of $\mathcal{F}_s$.

Ind from the times before s.

Ind of



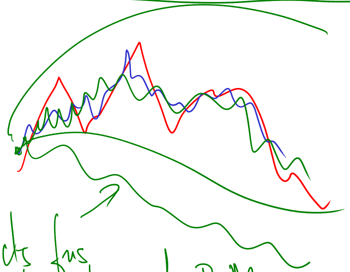
cts ⇒ No Jumps

5.2. Sample space, measure, and filtration.

- Discrete time: Sample space $\Omega = (\omega_1, \dots, \omega_N)$.

  ω_i = outcome of ith coin toss

- View $(\omega_1, \dots, \omega_N)$ as the trajectory of a random walk.
- Continuous time: Sample space $\Omega = C([0, \infty))$ (space of continuous functions).
  - ▷ It's infinite. No probability mass function!
  - ▷ Mathematically impossible to define $\boldsymbol{P}(A)$ for *all* $A \subseteq \Omega$.



View cts fns as trajectories of BM.

SRW corresponding to ω.

- Restrict our attention to $\mathcal{G}$, a subset of some sets $A \subseteq \Omega$, on which $\boldsymbol{P}$ can be defined.
  - ▷ $\mathcal{G}$ is a σ-algebra. (Closed countable under unions, complements, intersections.)
- $\boldsymbol{P}$ is called a *probability measure* on $(\Omega, \mathcal{G})$ if:
  - ▷ $\boldsymbol{P} \colon \mathcal{G} \to [0,1]$, $\boldsymbol{P}(\emptyset) = 0$, $\boldsymbol{P}(\Omega) = 1$.
  - ▷ $\boldsymbol{P}(A \cup B) = \boldsymbol{P}(A) + \boldsymbol{P}(B)$ if $A, B \in \mathcal{G}$ are disjoint.
  - ▷ If $A_n \in \mathcal{G}$, $\boldsymbol{P}\left(\bigcup_1^\infty A_n\right) = \lim_{n\to\infty} \boldsymbol{P}(A_n)$.

(i.e. $\forall A \in \mathcal{G}$, $P(A) \in [0,1]$)

- Random variables are *measurable* functions of the sample space:
  - ▷ Require $\{X \in A\} \in \mathcal{G}$ for every "nice" $A \subseteq \mathbb{R}$.
  - ▷ E.g. $\{X = 1\} \in \mathcal{G}$, $\{X > 5\} \in \mathcal{G}$, $\{X \in [3,4)\} \in \mathcal{G}$, etc.
  - ▷ Recall $\{X \in A\} = \{\omega \in \Omega \mid X(\omega) \in A\}$.

$\{\omega \in \Omega \mid X(\omega) > 0\} = \{X > 0\}$

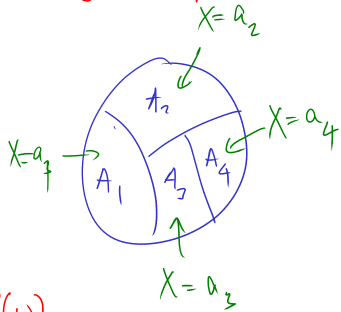
some subset of Ω.

- Expectation is a *Lebesgue Integral*: Notation $\boldsymbol{E}X = \int_\Omega X\, d\boldsymbol{P} = \int_\Omega X(\omega) d\boldsymbol{P}(\omega)$.
  - ▷ No simple formula.
  - ▷ If $X = \sum a_i \mathbf{1}_{A_i}$, then $\boldsymbol{E}X = \sum a_i \boldsymbol{P}(A_i)$. $(a_i \in \mathbb{R},\ A_i \in \mathcal{G}$ are disj$)$
  - ▷ $\mathbf{1}_A$ is the *indicator function* of A: $\mathbf{1}_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$

$$EX = \sum a_i P(X = a_i)$$

Discrete case: $EX = \sum_{\omega \in \Omega} X(\omega) P(\omega)$

$\omega \in \Omega$ $\quad$ $dP(\omega)$



almost surely: event of prob 1.

Proposition 5.3 (Useful properties of expectation).

(1) (Linearity) $\alpha, \beta \in \mathbb{R}$, X, Y *random variables*, $\boldsymbol{E}(\alpha X + \beta Y) = \alpha \boldsymbol{E} X + \beta \boldsymbol{E} Y$.

(2) (Positivity) *If* $X \geqslant 0$ *then* $\boldsymbol{E} X \geqslant 0$. *If* $X \geqslant 0$ *and* $\boldsymbol{E} X = 0$ *then* $X = 0$ *almost surely.*

$(P(X=0) = 1)$.

(3) (Layer Cake) *If* $X \geqslant 0$, $\boxed{\boldsymbol{E} X = \int_0^\infty \boldsymbol{P}(X \geqslant t)\, dt.}$

(4) *More generally, if* φ *is increasing,* $\varphi(0) = 0$ *then* $\boldsymbol{E} \varphi(X) = \int_0^\infty \varphi'(t)\, \boldsymbol{P}(X \geqslant t)\, dt.$

$(X \geqslant 0)$

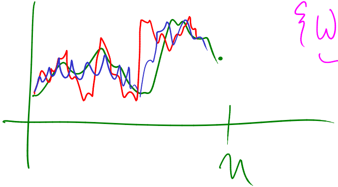
(5) (Unconscious Statistician Formula) *If PDF of* X *is* p, *then* $\boldsymbol{E} f(X) = \int_{-\infty}^{\infty} f(x) p(x)\, dx.$

(lazy)

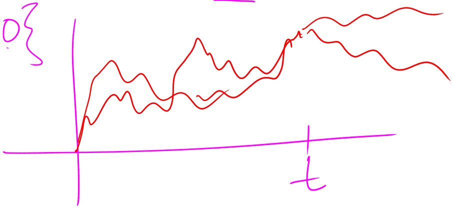
Know $EX = \int x\, p(x)\, dx$

$E f(X) = \int f(x)\, p(x)\, dx$

- Filtrations:
  - ▷ Discrete time: $\mathcal{F}_n$ = events described using the first n coin tosses.
  - ▷ Coin tosses doesn't translate well to continuous time.
  - ▷ Discrete time try #2: $\mathcal{F}_n$ = events described using the *trajectory* of the SRW up to time n.
  - ▷ Continuous time: $\mathcal{F}_t$ = events described using the *trajectory* of the *Brownian motion* up to time t.
  - ▷ If $t_i \leqslant t$, $A_i \subseteq \mathbb{R}$ then $\{W_{t_1} \in A_1, \dots, W_{t_n} \in A_n\} \in \mathcal{F}_t$. (Need all $t_i \leqslant t$!)
  - ▷ As before: if $s \leqslant t$, then $\mathcal{F}_s \subseteq \mathcal{F}_t$. ← Filtration.
  - ▷ Discrete time: $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Continuous time: $\mathcal{F}_0 = \{A \in \mathcal{G} \mid \boldsymbol{P}(A) \in \{0, 1\}\}$.



$\{W_{100} = 0\}$



fix $t \in \mathbb{R}$.

$A = [0, \infty)$

~~$A \subseteq \mathbb{R}$ nice~~

$\{ W_s \in \cancel{A} \}$ & $\underbrace{s \leq t}$

$\{ W_s > 0 \} \in \underline{\mathcal{F}_t}$? $\leftarrow$ Yes.

5.3. **Conditional expectation.**

- Notation $\boldsymbol{E}_t(X) = \boldsymbol{E}(X \mid \mathcal{F}_t)$ (read as conditional expectation of X given $\mathcal{F}_t$)
- No formula! But same intuition as discrete time.
- $\boldsymbol{E}_t X(\omega) =$ "average of X over $\Pi_t(\omega)$", where $\Pi_t(\omega) = \{\omega' \in \Omega \mid \omega'(s) = \omega(s)\ \forall s \leqslant t\}$.
- Mathematically problematic: $\boldsymbol{P}(\Pi_t(\omega)) = 0$ (but it still works out.)

Definition 5.4. $\boldsymbol{E}_t X$ is the unique *random variable* such that:

(1) $\boldsymbol{E}_t X$ is $\mathcal{F}_t$-measurable. (i.e $\forall A \subseteq \mathbb{R}$, $\{E_t X \in A\} \in \mathcal{F}_t$)

(2) For every $A \in \mathcal{F}_t$, $\displaystyle\int_A \boldsymbol{E}_t X \, d\boldsymbol{P} = \int_A X \, d\boldsymbol{P}$ (Disc time $\displaystyle\sum_{\omega \in A} E_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$)

Remark 5.5. Choosing $A = \Omega$ implies $\boldsymbol{E}(\boldsymbol{E}_t X) = \boldsymbol{E} X$.

Proposition 5.6 (Useful properties of conditional expectation)**.**

(1) If $\alpha, \beta \in \mathbb{R}$ are constants, X, Y, random variables $\boldsymbol{E}_t(\alpha X + \alpha Y) = \alpha \boldsymbol{E}_t X + \beta \boldsymbol{E}_t Y$.
(2) If $X \geqslant 0$, then $\boldsymbol{E}_t X \geqslant 0$. Equality holds if and only if $X = 0$ almost surely.
(3) (Tower property) *If $0 \leqslant s \leqslant t$, then $\boldsymbol{E}_s(\boldsymbol{E}_t X) = \boldsymbol{E}_s X$.*
(4) If X is $\mathcal{F}_t$ measurable, and Y is any random variable, then $\boldsymbol{E}_t(XY) = X \boldsymbol{E}_t Y$.
(5) If X is $\mathcal{F}_t$ measurable, then $\boldsymbol{E}_t X = X$ (follows by choosing $Y = 1$ above).
(6) If Y is independent *of $\mathcal{F}_t$, then $\boldsymbol{E}_t Y = \boldsymbol{E} Y$.*

Remark 5.7. These properties are exactly the same as in discrete time.

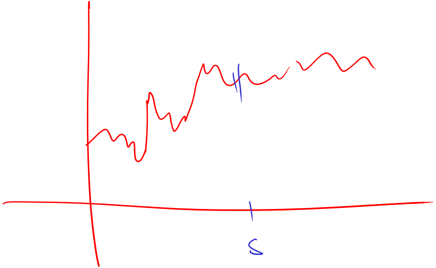
Last time: B. M.

$W_t \to$ B. M.

(0) W is a c/s process

(1) $W_t \sim N(0, t-s)$

& (2) $W_t - W_s$ is ind of $\mathcal{F}_s$



Definition 5.4. $\boldsymbol{E}_t X$ is the unique *random variable* such that:

$\left(E(X \mid \mathcal{F}_t) = E_t X \right.$ cond exp of X given $\left. \mathcal{F}_t \right)$

(1) $\boldsymbol{E}_t X$ is $\mathcal{F}_t$-measurable.

(2) For every $A \in \mathcal{F}_t$, $\cancel{\frac{1}{P(A)}}\int_A \boldsymbol{E}_t X \, d\boldsymbol{P} = \cancel{\frac{1}{P(A)}}\int_A X \, d\boldsymbol{P}$

Remark 5.5. Choosing $A = \Omega$ implies $\boldsymbol{E}(\boldsymbol{E}_t X) = \boldsymbol{E} X$.

Proposition 5.6 (Useful properties of conditional expectation).

(1) If $\alpha, \beta \in \mathbb{R}$ are constants, X, Y, random variables $\boldsymbol{E}_t(\alpha X + \alpha Y) = \alpha \boldsymbol{E}_t X + \beta \boldsymbol{E}_t Y$.

(2) If $X \geqslant 0$, then $\boldsymbol{E}_t X \geqslant 0$. Equality holds if and only if $X = 0$ almost surely.

(3) (Tower property) *If $0 \leqslant s \leqslant t$, then $\boldsymbol{E}_s(\boldsymbol{E}_t X) = \boldsymbol{E}_s X$.*

(4) If X is $\mathcal{F}_t$ measurable, and Y is any random variable, then $\boldsymbol{E}_t(XY) = X \boldsymbol{E}_t Y$.

(5) If X is $\mathcal{F}_t$ measurable, then $\boldsymbol{E}_t X = X$ (follows by choosing $Y = 1$ above).

(6) If Y is independent *of $\mathcal{F}_t$, then $\boldsymbol{E}_t Y = \boldsymbol{E} Y$.*

Remark 5.7. These properties are exactly the same as in discrete time.

Lemma 5.8 (Independence Lemma). *If X is $\mathcal{F}_t$ measurable, Y is independent of $\mathcal{F}_t$, and $f = f(x, y) \colon \mathbb{R}^2 \to \mathbb{R}$ is any function, then*

$$\boldsymbol{E}_t f(X, Y) = g(Y)\,, \qquad \text{where} \qquad g(y) = \boldsymbol{E} f(X, y)\,.$$

Remark 5.9. If ~~p_X is the PDF of X, then $\boldsymbol{E}_t f(X, Y) = \int_{\mathbb{R}} f(x, Y)\, p_X(x)\, dx$.~~

$p_Y \to$ PDF of Y

$\boldsymbol{E}_t f(X, Y) =$ "average Y & leave X alone"

$= \int_{\mathbb{R}} f(X, y)\, p_Y(y)\, dy$

5.4. **Martingales.**

Definition 5.10. An adapted process M is a martingale if for every $0 \leqslant s \leqslant t$, we have $\boldsymbol{E}_s M_t = M_s$.

Remark 5.11. As with discrete time, a martingale is a fair game: stopping based on information available today will not change your expected return.

Discrete time: M is a mg if $E_n M_{n+1} = M_n$.

tower $\implies \forall\, m \leq n,\ E_m M_n = M_m$

$E_m \to s$, $M_n \to t$, $M_m \to s$

$m, n \in \mathbb{N}$

$s, t \in [0, \infty)$.

Proposition 5.12. *Brownian motion is a martingale.*

Proof. $W \longrightarrow$ B. M.

WTS for every $s \le t$, $E_s W_t = W_s$

Note $E_s W_t = E_s (W_t - W_s + W_s)$

$= E_s (W_t - W_s) + E_s W_s$

$= E(W_t - W_s) + W_s$

($\because$ W_s is $\mathcal{F}_s$ meas &

$W_t - W_s$ is ind of $\mathcal{F}_s$)

$$= 0 + W_s \quad (\because W_t - W_s \sim N(0, t-s))$$

$$= W_s \quad \text{QED.}$$

6. Stochastic Integration

(Notation: Sometimes write $b_t = b(t)$)

6.1. Motivation.

at time t.

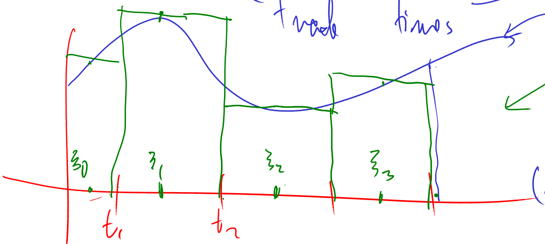
change in stock price

- Hold b_t shares of a stock with price S_t.
- Only trade at times $P = \{0 = t_1 < \dots, t_n = T\}$
- Net gain/loss from changes in stock price: $\displaystyle\sum_{k=0}^{n-1} b_{t_k} \Delta_k S$, where $\Delta_k S = S_{t_{k+1}} - S_{t_k}$.
- Trade continuously in time. Expect net gain/loss to be $\displaystyle\lim_{\|P\|\to 0} \sum_{k=0}^{n-1} b_{t_k} \Delta_k S = \int_0^T b_t \, dS_t$.
  - ▷ $\|P\| = \max_k (t_{k+1} - t_k)$. (Norm (P) $\approx$ mesh size(P))
  - ▷ Riemann-Stieltjes integral: $\displaystyle\lim_{\|P\|\to 0} \sum_{k=0}^{n-1} b_{\xi_k} \Delta_k S = \int_0^T b_t \, dS_t$,
  - ▷ The $\xi_k \in [t_k, t_{k+1}]$ can be chosen arbitrarily.
  - ▷ Only works if the *first variation* of S is finite. False for most stochastic processes.

$t_0 = 0 \quad t_1 \quad t_2 \quad \cdots \quad t_{n-1} \quad T = t_n$

trade times

Some fn f.

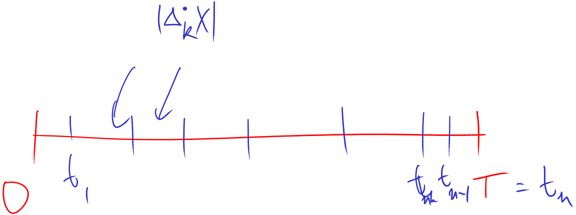


$$\lim_{\|P\| \to 0} \sum f(\xi_k)(t_{k+1} - t_k)$$

$$(\text{Riem int}) = \int_0^T f(t)\, dt$$

Riemann-Stieltjes int:

$$\underbrace{\lim_{\|P\|\to 0} \sum f(\xi_k)\left(S_{t_{k+1}} - S_{t_k}\right)}_{\int_0^T f(t)\, dS_t}$$



$$\Delta_k X = X_{t_{k+1}} - X_{t_k} = \text{inc of } X \text{ over } [t_k, t_{k+1}]$$

6.2. First Variation.

Definition 6.1. For any process X, define the *first variation* by

$$V_{[0,T]}(X) \stackrel{\text{def}}{=} \lim_{\|P\|\to 0} \sum_{k=0}^{n-1} |\Delta_k X| \,.\stackrel{\text{def}}{=} \lim_{\|P\|\to 0} \sum_{k=0}^{n-1} |X_{t_{k+1}} - X_{t_k}| \,.$$

Remark 6.2. If $X(t)$ is a differentiable function of t then $V_{[0,T]}X < \infty$.

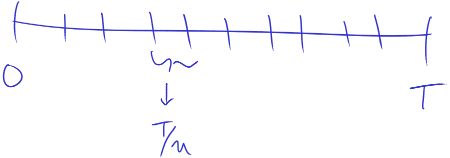
Proposition 6.3. $\boldsymbol{E} V_{[0,T]} W = \infty$

Remark 6.4. In fact, $V_{[0,T]}W = \infty$ almost surely. Brownian motion does not have finite first variation.

Remark 6.5. The Riemann-Stieltjes integral $\int_0^T b_t \, dW_t$ *does not exist.*

Let's check Prop 6.3:

N some large #.



Let $t_k = \frac{k}{n}$

$$E\, V_{[0,T]} W = E \lim_{n\to\infty} \sum \left| W_{t_{k+1}} - W_{t_k} \right| = \lim_{N\to\infty} \sum \underbrace{E\left| W_{\frac{k+1}{n}} - W_{\frac{k}{n}} \right|}.$$

Know $W_{\frac{k+1}{n}} - W_{\frac{k}{n}} \sim N\left(0, \frac{1}{n}\right)$

$$\Rightarrow E\left|W_{\frac{k+1}{n}} - W_{\frac{k}{n}}\right| = \underset{\substack{\uparrow \\ \text{some constant}}}{c} \cdot \left(\frac{1}{\sqrt{n}}\right)$$

$$\Rightarrow E\sum_{k=0}^{n-1}\left|W_{\frac{k+1}{n}} - W_{\frac{k}{n}}\right| = c\sum_{k=0}^{n-1}\frac{1}{\sqrt{n}} = c\sqrt{n} \xrightarrow{n\to\infty} \infty$$

(Note: If $X \sim N(0, \sigma^2)$,

then $E|X| = \displaystyle\int_{-\infty}^{\infty} |x|\, e^{-x^2/2\sigma^2} \frac{dx}{\sqrt{2\pi}\,\sigma}$

Put $y = \frac{x}{\sigma}$

$dx = \sigma\, dy$

$$= \int_{-\infty}^{\infty} |y| \sigma \; e^{-y^2/2} \frac{\cancel{\sigma}\, dy}{\sqrt{2\pi}\, \cancel{\sigma}}$$

$$= \sigma \underbrace{\int_{-\infty}^{\infty} |y|\, e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}}_{c}$$

$$\Rightarrow E|N(0,\sigma^2)| = \sigma \cdot (c)$$

6.3. **Quadratic Variation.**

Definition 6.6. If M is a continuous time adapted process, define

$$[M, M]_T = \lim_{\|P\| \to 0} \sum_{k=0}^{n-1} (M_{t_{k+1}} - M_{t_k})^2 = \lim_{\|P\| \to 0} \sum_{k=0}^{n-1} (\Delta_k M)^2 .$$

Proposition 6.7. *For continuous processes the following hold:*

1. *Finite first variation implies the quadratic variation is* 0
2. *Finite (non-zero) quadratic variation implies the first variation is infinite.*

(Will revisit this shortly)

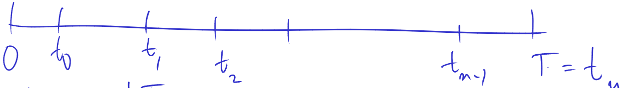
$[M, M]_T$ = Q.V. of M up to T

M adapted $\Rightarrow$ $[M, M]$ is an adapted (inc) process.

Proposition 6.8. $\underbrace{[W,W]_T}$ $= T$ *almost surely.*

Remark 6.9. For use in the proof: $\mathrm{Var}(\mathcal{N}(0,\sigma^2)^2) = \underbrace{\boldsymbol{E}\mathcal{N}(0,\sigma^2)^4}_{3\sigma^4} - (\boldsymbol{E}\underbrace{\mathcal{N}(0,\sigma^2)^2}_{\sigma^4})^2 = 2\sigma^4.$

Proof:.

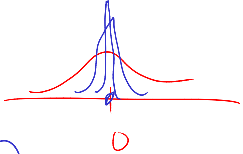


Set $t_k = \dfrac{kT}{n}$

T/n.

QV: Want $\lim_{N\to\infty} \left[\underbrace{\sum_{0}^{n-1} (\Delta_k W)^2}_{} - T \right] = 0$

(Will imply $[W,W]_T = T$).



Will show ① $E\left[\sum_{0}^{n-1} (\Delta_k W)^2 - T\right] = 0$

& ② $\mathrm{Var}\left(\sum_{0}^{n-1} (\Delta_k W)^2 - T\right) \xrightarrow{N \to \infty} 0$

①+② $\Rightarrow \lim_{N \to \infty} \sum (\Delta_k W)^2 = T$ (i.e $[W,W]_T = T$).

Pf of ①: $E\left(\sum_{0}^{n-1} (\Delta_k W)^2 - T\right) =$

$$= \sum_{0}^{n-1} \frac{T}{n} - T = 0 \qquad \Delta_k W \sim N\left(0, \frac{T}{n}\right)$$

Pf of (2):
$$\mathrm{Var}\left(\sum_{0}^{n-1} (\Delta_k W)^2 - T\right) = \mathrm{Var}\left(\sum_{0}^{n-1} (\Delta_k W)^2\right)$$

$$= \sum_{0}^{n-1} \mathrm{Var}\left[(\Delta_k W)^2\right] \qquad \left((\Delta_k W)^2 \sim N\left(0, \frac{T}{n}\right)^2\right.$$

$$\Rightarrow \mathrm{Var}\left((\Delta_k W)^2\right) = 2 \frac{T^2}{n^2}.)$$

$$= \sum_{0}^{n-1} 2 \frac{T^2}{n^2} = \frac{2T^2}{n} \xrightarrow{n \to \infty} 0$$

QED.

Proposition 6.10. $\underbrace{W_t^2}_{} - \underbrace{[W,W]_t}_{}$ *is a martingale.*

Pf: Know $[W, W]_t = t$.

$$\Rightarrow W_t^2 - [W,W]_t = W_t^2 - t$$

Let $M_t = W_t^2 - t$. NTS M is a mg

i.e. NTS $E_s(M_t) = M_s$

i.e. NTS $E_s(W_t^2 - t) = W_s^2 - s$

Pf: $E_s\left((W_t - W_s + W_s)^2 - t\right)$

$$= E_s\left(\underline{(W_t - W_s)^2} + W_s^2 + 2W_s(W_t - W_s)\right) - t$$

$$= E(W_t - W_s)^2 + W_s^2 + E_s[W_s(W_t - W_s)] - t$$

$$= t - s + W_s^2 + W_s\, \underbrace{E(W_t - W_s)}_{0} - t$$

(W_s is $\mathcal{F}_s$ meas)

($\because \underline{W_t - W_s}$ is ind of $\mathcal{F}_s$)

$$= t - s + W_s^2 - t = W_s^2 - s$$

//

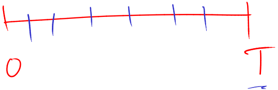
Theorem 6.11. *Let M be a continuous martingale.*

1. *(1) $\boldsymbol{E}M_t^2 < \infty$ if and only if $\boldsymbol{E}[M,M]_t < \infty$.*
2. *(2) In this case $M_t^2 - [M,M]$ is a continuous martingale.*
3. *(3) Conversely, if $M_t^2 - A_t$ is a martingale for any continuous, increasing process A such that $A_0 = 0$, then we must have $A_t = [M,M]_t$.*

Remark 6.12. The optional problem on HW2 gives some intuition in discrete time.

Last time:

1st Var: $V_{[0,T]} X = \lim_{\|P\| \to 0} \sum_{i=1}^{n} |\Delta_i X|$



$P = \{0 = t_0 < t_1 < \dots t_n = T\}$

$\|P\| = \max_i \; t_{i+1} - t_i$

$\Delta_i X = X_{t_{i+1}} - X_{t_i}$

B.M does not have finite 1st Var.

$V_{[0,T]} W = +\infty \quad (a.s).$

Need $V_{[0,T]} X < \infty$

Quadratic Var:

$$[X,X]_T = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} (\Delta_i X)^2$$

in order to define Riemann Int

Saw $[W,W]_T = T$ (a.s.)

& $W_t^2 - [W,W]_t = W_t^2 - t$ is a mg.

Theorem 6.11. *Let M be a continuous martingale.*

(1) $\boldsymbol{E}M_t^2 < \infty$ *if and only if* $\boldsymbol{E}[M,M]_t < \infty$.
(2) *In this case* $M_t^2 - [M,M]_t$ *is a continuous martingale.*
(3) *Conversely, if* $M_t^2 - A_t$ *is a martingale for any continuous, increasing process* A *such that* $A_0 = 0$*, then we must have* $A_t = [M,M]_t$.

Remark 6.12. The optional problem on HW2 gives some intuition in discrete time.

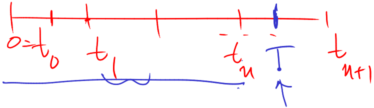
Remark 6.13. If X has finite first variation, then $|X_{t+\delta t} - X_t| \approx O(\delta t)$.

Remark 6.14. If X has finite quadratic variation, then $|X_{t+\delta t} - X_t| \approx O(\sqrt{\delta t}) \gg O(\delta t)$.

Intuition: Finite 1st var $\longrightarrow$ "differentiable in time"

$\longrightarrow$ Finite (non-zero) QV $\longrightarrow$ Never diff

6.4. Itô Integrals.



- $D_t = D(t)$ some adapted process (position on an asset).
- $P = \{0 = t_0 < t_1 < \cdots\}$ increasing sequence of times.
- $\|P\| = \max_i (t_{i+1} - t_i)$, and $\Delta_i X = X_{t_{i+1}} - X_{t_i}$.
- W : standard Brownian motion.
- $I_P(T) \stackrel{\text{def}}{=} \sum_{i=0}^{n-1} D_{t_i} \Delta_i W + D_{t_n}(W_T - W_{t_n})$

if $T \in [t_n, t_{n+1})$

Definition 6.15. The *Itô Integral* of D with respect to Brownian motion is defined by

$$\int_0^T D_s\, dW_s = I_T = \int_0^T D_t\, dW_t = \lim_{\|P\| \to 0} I_P(T) .$$

Only works because D_t is an adapted process.

Remark 6.16. Suppose for simplicity $T = t_n$.

(1) Riemann integrals: $\lim_{\|P\| \to 0} \sum D_{\xi_i} \Delta_i W$ exists, for any $\xi_i \in [t_i, t_{i+1}]$.

(2) Itô integrals: Need $\xi_i = t_i$ for the limit to exist.

& Need D to be adapted.

Theorem 6.17. *If* $\boldsymbol{E} \int_0^T D_t^2 \, dt < \infty$ ~~*a.s.*~~, *then:*

(1) $I_T = \lim_{\|P\| \to 0} I_P(T)$ *exists a.s., and* $\boldsymbol{E} I(T)^2 < \infty$. $\left(I_T = \int_0^T D_t \, dW_t \right)$

(2) *The process* I_T *is a martingale:* $\boldsymbol{E}_s I_t = \boldsymbol{E}_s \int_0^t D_r \, dW_r = \int_0^s D_r \, dW_r = I_s$

(3) $[I, I]_T = \int_0^T D_t^2 \, dt$ *a.s.*   (Note $\int_0^T D_t^2 \, dt$ is a std Rieman Int)

Remark 6.18. If we only had $\int_0^T D_t^2 \, dt < \infty$ a.s., then $I(T) = \lim_{\|P\| \to 0} I_P(T)$ still exists, and is finite a.s. But it may not be a martingale (it's a *local martingale*).

NOTATION: $EX^2 = E(X^2)$ NOT $(EX)^2$

Corollary 6.19 (Itô isometry). $\boldsymbol{E}\left(\int_0^T D_t\, dW_t\right)^2 = \boldsymbol{E}\int_0^T D_t^2\, dt = \int_0^T E D_t^2\, dt$

Proof.

Re Note For Riemann Integrals,

$$E\int_0^T D_t^2\, dt = \int_0^T E D_t^2\, dt$$

NOT dW
Riemann Int

Intuition : $E \int_0^T D_t^2 \, dt$ (Riemann) $= E \lim_{\|P\| \to 0} \sum D_{t_i}^2 (t_{i+1} - t_i)$

$$= \lim_{\|P\| \to 0} E \sum D_{t_i}^2 (t_{i+1} - t_i)$$

$$= \lim_{\|P\| \to 0} \sum \left(E D_{t_i}^2 \right) (t_{i+1} - t_i)$$

$$= \int_0^T \left(E D_t^2 \right) dt$$

a

Pf of Itô isom (Assuming prop of Ito int):

Know $I_t = \int_0^t D_s \, dW_s$ is a mg

& $[I,I]_t = \int_0^t D_s^2 \, ds$

$\Rightarrow I_t^2 - [I,I]_t$ is a mg!

$\Rightarrow E\left(I_t^2 - [I,I]_t\right) = E\left(I_0^2 - [I,I]_0\right) = 0$

$$\Rightarrow \quad E I_t^2 = E[I,I]_t$$

$$\Rightarrow \quad E\left(\int_0^t D_s \, dW_s\right)^2 = E\left(\int_0^t D_s^2 \, ds\right)$$

//.

Intuition for Theorem 6.17 (2). Check $I_P(T)$ is a martingale.

$$I_P(T) = \sum_{i=0}^{n-1} D_{t_i} \Delta_i W + D_{t_n}(W_T - W_{t_n}) \quad \text{if } T \in [t_n, t_{n+1})$$

NTS $E_s I_P(t) = I_P(s)$

for simplicity suppose $s = t_m$ & $t = t_n$, $m \leq n$.

$$I_P(s) = I_P(t_m) = \sum_{i=0}^{m-1} D_{t_i} \Delta_i W$$

$$I_P(t) = \underline{I_P(t_n)} = \sum_{i=0}^{n-1} D_{t_i} \Delta_i W$$

$$\Rightarrow E_s(\ ) = E_{t_m}\left(\sum_{i=0}^{n-1} D_{t_i} \Delta_i W\right)$$

$$= \underbrace{E_{t_m}\left(\sum_{i=0}^{m-1} D_{t_i} \Delta_i W\right)}_{\mathcal{F}_{t_m}\text{-mea } (\because t_i < t_m)} + E_{t_m}\left(\sum_{i=m}^{n-1} D_{t_i} \left(W_{t_{i+1}} - W_{t_i}\right)\right)$$

$$= \underbrace{\sum_{i=0}^{m-1} D_{t_i} \Delta_i W}_{I_p(s)} + \sum_{i=m}^{n-1} E_{t_m} E_{t_i} \left[D_{t_i} \left(W_{t_{i+1}} - W_{t_i} \right) \right]$$

$$= I_p(s) + \sum_{i=m}^{n-1} E_{t_m} \left(D_{t_i} \, E_{t_i} \left(W_{t_{i+1}} - W_{t_i} \right) \right)$$

($\because D_{t_i}$ is $\mathcal{F}_{t_i}$-meas)

$$= I_p(s) + \sum_{i=m}^{n-1} E_{t_m}\Big(D_{t_i} \underbrace{E\left(W_{t_{i+1}} - W_{t_i}\right)}_{\downarrow\ 0} \Big)$$

($\because W_{t_{i+1}} - W_{t_i}$ is ind of $\mathcal{F}_i$)

& $W_{t_{i+1}} - W_{t_i} \sim N(0, t_{i+1} - t_i)$)

$$= I_p(s)$$

QED.

Intuition: Itô Isom:
$$\mathbb{E}\left(\int_0^T D_s\, dW_s\right)^2 = \mathbb{E}\int_0^T D_s^2\, ds$$

Let's check by hand:
$$\mathbb{E}\left(\sum_{i=0}^{n-1} D_{t_i}\, \Delta_i W\right)^2 = \underbrace{\mathbb{E}\sum_{i=0}^{n-1} D_{t_i}^2 (t_{i+1}-t_i)}_{1}$$

Expand LHS:
$$\mathbb{E}\left(\sum_{i=0}^{n-1} D_{t_i}\, \Delta_i W\right)^2 = \mathbb{E}\left(\sum_{i=0}^{n-1} D_{t_i}^2 (\Delta_i W)^2 + \right.$$

$$\underbrace{E\, 2 \sum_{j=0}^{n-1} \sum_{i=0}^{j-1} D_{t_i} \Delta_i W \; D_{t_j} \Delta_j W}_{2}$$

1

$$1 = \sum_{i=0}^{n-1} E D_{t_i}^2 (\Delta_i W)^2 = \sum_{i=0}^{n-1} E\, E_{t_i}\left(D_{t_i}^2 \left(W_{t_{i+1}} - W_{t_i}\right)^2 \right)$$

$$= \sum_{i=0}^{n-1} E D_{t_i}^2 \; E\left(W_{t_{i+1}} - W_{t_i}\right)^2$$

$$= \sum_{i=0}^{n-1} E D_{t_i}^2 (t_{i+1} - t_i) = \text{Desired RHS.}$$

$$(2) = 2 \sum_{j=0}^{n-1} \sum_{i=0}^{j-1} E\left(D_{t_i}(W_{t_{i+1}} - W_{t_i}) D_{t_j} (W_{t_{j+1}} - W_{t_j}) \right)$$

$$= 2 \sum_{j=0}^{n-1} \sum_{i=0}^{j-1} E \, E_{t_j}\left(D_{t_i}(W_{t_{i+1}} - W_{t_i}) \; D_{t_j} (W_{t_{j+1}} - W_{t_j}) \right)$$

Note $i < j \Rightarrow D_{t_i}, W_{t_{i+1}}, W_{t_i}, D_{t_j}$ meas

$$= 2 \sum_{j=0}^{n-1} \sum_{i=0}^{j-1} E\left(D_{t_i} (W_{t_{i+1}} - W_{t_i}) D_{t_j} \underbrace{E(W_{t_{j+1}} - W_{t_j})}_{0} \right)$$

$$= 0$$

QED.

Proposition 6.20. *If $\alpha, \tilde{\alpha} \in \mathbb{R}$, $D, \tilde{D}$ adapted processes*

$$\int_0^T (\alpha D_s + \tilde{\alpha}\tilde{D}_s)\, dW_s = \alpha \int_0^T D_s\, dW_s + \tilde{\alpha} \int_0^T \tilde{D}_s\, dW_s$$

Proposition 6.21. $\int_0^{T_1} D_s\, dW_s + \int_{T_1}^{T_2} D_s\, dW_s = \int_0^{T_2} D_s\, dW_s$

Question 6.22. *If $D \geqslant 0$, then must $\int_0^T D_t\, dW_t \geqslant 0$?* ← False!

Intuition: $\int_0^T (\alpha D_s + \tilde{\alpha}\tilde{D}_s)\, dW_s = \lim \sum (\alpha D_{t_i} + \tilde{\alpha}\tilde{D}_{t_i}) \Delta_i W$

$\lim \sum \alpha D_{t_i} \Delta_i W + \tilde{\alpha} \lim \sum \tilde{D}_{t_i} \Delta_i W$

6.5. **Semi-martingales and Itô Processes.**

Question 6.23. *What is* $\int_0^t W_s \, dW_s$*?*

Definition 6.24. A *semi-martingale* is a process of the form $X = X_0 + B + M$ where:

- ▷ X_0 is $\mathcal{F}_0$-measurable (typically X_0 is constant).
- ▷ B is an adapted process with finite first variation. (aka Bounded Variation)
- ▷ M is a martingale.

Definition 6.25. An *Itô-process* is a semi-martingale $X = X_0 + B + M$, where:

- ▷ $B_t = \int_0^t b_s \, ds$, with $\int_0^t |b_s| \, ds < \infty$ (Std Riemann int) $\rightarrow dB_t = b_t \, dt$
- ▷ $M_t = \int_0^t \sigma_s \, dW_s$, with $\int_0^t |\sigma_s|^2 \, ds < \infty$ (Itô int) $\rightarrow dM_t = \sigma_t \, dW_t$

Remark 6.26. Short hand notation for Itô processes: $dX_t = b_t \, dt + \sigma_t \, dW_t$.

Remark 6.27. Expressing $X = X_0 + B + M$ (or $dX = b \, dt + \sigma \, dW$) is called the *semi-martingale decomposition* or the *Itô decomposition* of X.

Theorem 6.28 (Itô formula)**.** *If* $f \in C^{1,2}$, *then*

$$df(t, X_t) = \partial_t f(t, X_t)\, dt + \partial_x f(t, X_t)\, dX_t + \boxed{\frac{1}{2}\partial_x^2 f(t, X_t)\, d[X, X]_t}$$

Remark 6.29*.* This is the main tool we will use going forward. We will return and study it thoroughly after understanding all the notions involved.

Proposition 6.30. *If* $X = X_0 + B + M$, *then* $[X, X] = [M, M]$.

Proposition 6.31 (Uniqueness)**.** *The Itô decomposition is unique. That is, if* $X = X_0 + B + M = Y_0 + C + N$*, with:*

- ▷ B, C *bounded variation,* $B_0 = C_0 = 0$
- ▷ M, N *martingale,* $M_0 = N_0 = 0$.

Then $X_0 = Y_0$*,* $B = C$ *and* $M = N$*.*

last time

Definition 6.24. A *semi-martingale* is a process of the form $X = X_0 + B + M$ where:

- X_0 is $\mathcal{F}_0$-measurable (typically X_0 is constant).
- B is an adapted process with finite first variation. (bdd variation)
- M is a martingale.

Definition 6.25. An *Itô-process* is a semi-martingale $X = X_0 + B + M$, where:

- $B_t = \int_0^t b_s\, ds$, with $\int_0^t |b_s|\, ds < \infty$ (Rieman Int)
- $M_t = \int_0^t \sigma_s\, dW_s$, with $\int_0^t |\sigma_s|^2\, ds < \infty$ (Itô int)

dB dM

Remark 6.26. Short hand notation for Itô processes: $dX_t = b_t\, dt + \sigma_t\, dW_t$.

Remark 6.27. Expressing $X = X_0 + B + M$ (or $dX = b\, dt + \sigma\, dW$) is called the *semi-martingale decomposition* or the *Itô decomposition* of X.

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Remark 6.29. This is the main tool we will use going forward. We will return and study it thoroughly after understanding all the notions involved.

More today

Proposition 6.30. *If* $X = X_0 + B + M$*, then* $[X, X] = [M, M]$.

(M, B are both cts)

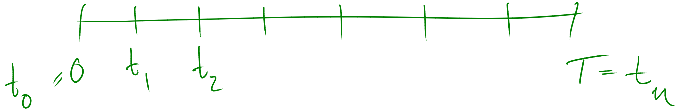
Note: $dM = \sigma \, dW$ then $M_t = \cancel{M_0}^{0} + \int_0^t \sigma_s \, dW_s$

Know $[M, M]_t = \int_0^t \sigma_s^2 \, ds$

($\Leftrightarrow$ $d[M, M]_t = \sigma_t^2 \, dt$ (short hand notation).)

Alt Notation: $d[M,M]_t = dM_t \, dM_t$

Intuition:



$$[X,X]_T = \lim_{\|P\|\to 0} \sum (\Delta_i X)^2 \qquad (\Delta_i X = X_{t_{i+1}} - X_{t_i})$$

$$= \lim_{\|P\|\to 0} \sum (\Delta_i B + \Delta_i M)^2$$

$$= \lim_{\|P\|\to 0} \sum \underbrace{(\Delta_i B)^2}_{\xrightarrow[\|P\|\to 0]{\text{IOU.}} 0} + \underbrace{(\Delta_i M)^2}_{\xrightarrow{\|P\|\to 0} [M,M]_T} + \underbrace{2(\Delta_i B)(\Delta_i M)}_{\xrightarrow[\|P\|\to 0]{\text{IOU}} 0}$$

① Compute $\lim_{\|P\|\to 0} \sum (\Delta_i B)^2 \leq \lim_{\|P\|\to 0} \left(\underbrace{\max_i |\Delta_i B|} \ \underbrace{\sum |\Delta_i B|} \right)$

$$\lim_{\|P\|\to 0} \max_i |B_{t_{i+1}} - B_{t_i}| = 0$$

($\because$ B is cts)

$\|P\| \to 0$

$V_{[0,T]}^{B}$

(1st Var B) $< \infty$.

$= 0$

② $$\lim_{\|P\|\to 0} \cancel{X} \sum (\Delta_i B)(\Delta_i M) \underset{\text{Cauchy Schwarz}}{\le} \lim_{\|P\|\to 0} \underbrace{\underbrace{\left(\sum |\Delta_i B|^2\right)^{1/2}}_{\xrightarrow{} 0} \underbrace{\left(\sum |\Delta_i M|^2\right)^{1/2}}_{\xrightarrow{\|P\|\to 0} [M,M]}}_{0}$$

$$\left(\underbrace{\sum x_i y_i}_{x \cdot y} \le \underbrace{\left(\sum x_i^2\right)^{1/2}}_{\|x\|} \underbrace{\left(\sum y_i^2\right)^{1/2}}_{\|y\|}. \right)$$

Proposition 6.31 (Uniqueness)**.** *The Itô decomposition is unique. That is, if* $X = X_0 + B + M = Y_0 + C + N$*, with:*

- ▷ B, C *bounded variation,* $B_0 = C_0 = 0$
- ▷ M, N *martingale,* $M_0 = N_0 = 0.$

Then $X_0 = Y_0$*,* $B = C$ *and* $M = N$*.*

Check: ① At $t = 0$, $M_0 = N_0 = 0$, $B_0 = C_0 = 0$

$$\Rightarrow X_0 = Y_0.$$

② Know $B + M = C + N$

$$\Rightarrow B - C = N - M$$

Finite 1st var

(B.V.)

Mg

Know $E(N_t - M_t)^2 = E[N-M, N-M]_t$

$= E[B-C, B-C]_t = 0$

$\Rightarrow N = M$ & $B = C$

Corollary 6.32. *Let* $dX_t = \overbrace{\boxed{b_t\,dt} + \sigma_t\,dW_t}$ *with* $\boldsymbol{E}\int_0^t b_s\,ds < \infty$ *and* $\boldsymbol{E}\int_0^t \sigma_s^2\,ds < \infty$. *Then* X *is a martingale if and only if* $b = 0$.

(If X was a mg, then

$$X_t = X_0 + \underbrace{0}_{BV} + \overbrace{(X_t - X_0)}^{Mg}$$

$$= X_0 + \int_0^t b_s\,ds + \int_0^t \sigma_s\,dW_s$$

Uniqueness $\Rightarrow \int_0^t b_s\,ds = 0 \ \forall t$ & $X_t = X_0 + \int_0^t \sigma_s\,dW_s$ $(\Rightarrow b = 0)$

Definition 6.33. If $dX = b\,dt + \boxed{\sigma\,dW}$, define $\displaystyle\int_0^T D_t\,dX_t = \int_0^T D_t b_t\,dt + \int_0^T D_t\sigma_t\,dW_t$.

Remark 6.34. Note $\displaystyle\int_0^T D_t b_t\,dt$ is a *Riemann integral*, and $\displaystyle\int_0^T D_t\sigma_t\,dW_t$ is a *Itô integral*.

6.6. **Itô's formula.**

Remark 6.35. If f and X are differentiable, then

$$df(t, X_t) = \partial_t f(t, X_t)\, dt + \partial_x f(t, X_t)\, dX_t$$

Chain Rule: $f = f(t, x)$ diff.

Want $x = x(t)$ some diff fn.

What is $\frac{d}{dt}\left[f(t, x(t)) \right]$ $\underset{\text{Chain Rule}}{=}$ $\partial_t f(t, x(t)) + \partial_x f(t, x(t)) \frac{dx}{dt}$

$x(t) \longrightarrow X_t$

$Y_t = f(t, X_t) \rightarrow$ some process

$$dY_t = \left(\frac{d}{dt} f(t, X_t)\right) dt$$

$$= \partial_t f(t, X_t)\, dt + \partial_x f(t, X_t) \frac{dX_t}{\cancel{dt}} \cancel{dt}$$

$$d\left(f(t, X_t)\right) = \underline{\partial_t f(t, X_t)}\, dt + \underbrace{\partial_x f(t, X_t)\, dX_t}$$

ONLY WORKS IF X is a diff fn of t.

__All__ (non-constant) Mg's are __NOT__ diff fn's of t.

Itô – Doeblin

Theorem (Itô's formula, Theorem 6.28)**.** *If $f \in C^{1,2}$, then*

$$df(t, X_t) = \partial_t f(t, X_t)\, dt + \partial_x f(t, X_t)\, dX_t + \frac{1}{2}\partial_x^2 f(t, X_t)\, d[X, X]_t$$

Remark 6.36. If $dX_t = b_t\, dt + \sigma_t\, dW_t$ then

$$\cancel{df(t, X_t) = \left(\partial_t f(t, X_t) + b_t + \frac{1}{2}\sigma_t^2\right) dt + \partial_x f(t, X_t)\sigma_t\, dW_t}.$$

$f \in C^{1,2} \longrightarrow f = f(t, x)$

① f is diff in t

② f is twice diff in x.

$Y_t = f(t, X_t) \longleftarrow$ new process.

Q: $dY_t = ?$

Note $dX = b\,dt + \sigma\,dW$

$$d f(t, X_t) = \partial_t f(t, X_t)\,dt + \partial_x f(t, X_t)\,dX + \frac{1}{2}\partial_x^2 f(t, X_t)\,d[X,X]$$

$$= \partial_t f\,dt + \partial_x f\,(b\,dt + \sigma\,dW) + \frac{1}{2}\partial_x^2 f\,\sigma^2\,dt$$

$$= \left(\partial_t f + \partial_x f \cdot b + \frac{1}{2}\partial_x^2 f\,\sigma^2\right)dt + \partial_x f\,\sigma\,dW$$

Intuition behind Itô's formula?

Simple case: $f(t,x) = f(x)$ (ind of t).

$$X_t = W_t.$$

Itô: $d f(X_t) = d f(W_t) = f'(W_t)\,dt + \frac{1}{2} f''(W_t)\,dt$

Will show

$$f(W_T) - f(W_0) = \sum \Delta_i f(W) = \sum f(W_{t_{i+1}}) - f(W_{t_i})$$

Taylor Exp

$$= \underbrace{\sum f'(W_{t_i})(W_{t_{i+1}} - W_{t_i})}_{1} + \underbrace{\frac{1}{2}\sum f''(W_{t_i})(W_{t_{i+1}} - W_{t_i})^2}_{2} + \underbrace{\text{small terms}}_{\longrightarrow 0}$$

a

$$\lim_{\|P\|\to 0} (1) = \lim_{\|P\|\to 0} \sum f'(W_{t_i})\,\Delta_i W \longrightarrow \int_0^T f'(W_t)\,dW_t$$

$$\lim_{\|P\|\to 0} (2) = \lim_{\|P\|\to 0} \frac{1}{2}\sum f''(W_{t_i})\,(\Delta_i W)^2$$

$$= \lim_{\|P\|\to 0} \frac{1}{2}\sum f''(W_{t_i})\,(t_{i+1} - t_i) \xrightarrow{\|P\|\to 0} \frac{1}{2}\int_0^T f''(W_t)\,dt$$

$$+\lim_{\|P\|\to 0} \frac{1}{2}\sum \underbrace{f''(W_{t_i})\cdot\left((\Delta_i W)^2 - (t_{i+1}-t_i)\right)}$$

NTS $\longrightarrow 0$

Note : $(\Delta_i W)^2 - (t_{i+1}-t_i) \sim \underbrace{\left(N(0, t_{i+1}-t_i)^2 - (t_{i+1}-t_i)\right)}$

1. Mean 0.
2. Variance $2(\underline{t_{i+1}-t_i})^2$

Guess: $\frac{1}{2} \sum f''(W_{t_i}) \left((\Delta_i W)^2 - (t_{i+1} - t_i) \right)$

mean 0 & variance $\underbrace{\sum f''(W_{t_i})^2 \cdot 2 (t_{i+1} - t_i)^2}$ $\xrightarrow[|P| \to 0]{} 0$

Example 6.37. $\boxed{\text{Find the quadratic variation of } W_t^2.}$

Guess

Q: ~~Q4~~ What is QV of $2W_t$? $\longrightarrow$ $4t$. ✓

Guess

What is QV of tW_t? $\longrightarrow$ t^3 ✓

What is QV of $W_t W_t$ $\longrightarrow$ Guess $W_t^2 t$

Hopelessly Wrong!!

QV of W:

Let $X_t = W_t$.

Let $f(t, x) = x^2$

Then $W_t^2 = f(t, W_t)$.

By Ito: $d(W_t^2) = \partial_t f(t, X_t)\, dt + \partial_x f(t, X_t)\, dX_t$

$+ \frac{1}{2} \partial_x^2 f(t, X_t)\, d[X, X]$.

① $\partial_t f = 0$

② $\partial_x f = 2x$

③ $\partial_x^2 f = 2$

④ $d[X,X] = dt$

$$= 0 + 2X_t\, dX_t + \frac{1}{2}\cdot 2 \cdot dt$$

$$= 2W_t\, dW_t + dt$$

$$\Rightarrow d(W_t^2) = 2W_t\, dW_t + \underline{dt}$$

$$\Rightarrow d\left[W^2, W^2\right]_t = 4W_t^2\, dt + 0$$

$$\Rightarrow [W, W]_T = \int_0^T 4W_t^2\, dt$$

Example 6.38. Find $\int_0^t W_s \, dW_s$. ← Shulam's Recitation.

Example 6.39. Let $M_t = W_t$, and $N_t = W_t^2 - t$.

- ▷ We know M, N are martingales.
- ▷ Is MN a martingale?

Is $W_t (W_t^2 - t)$ a mg?

$\iff$ Is $W_t^3 - W_t t$ a mg?

Option 1: Find $E_s(W_t^3 - W_t t)$ by Writing $W_t = W_t - W_s + W_s$ & ind lemma

Option 2: Itô's formula:

$$Y_t = f(t, W_t), \quad \text{where } f(t,x) = x^3 - xt.$$

Compute $dY \overset{\text{Itô}}{=} = (\sim\sim)\,dt + (\sim\sim)\,dW$

Y is a mg if and only if the dt term vanishes.

① $\partial_t f = -x$

② $\partial_x f = 3x^2 - t$

③ $\partial_x^2 f = 6x$

④ $d[W,W]_t = dt$

$$dY = d\,f(t, \underbrace{W_t})$$

$$= \partial_t f \, dt + \partial_x f \, dW + \frac{1}{2}\partial_x^2 f \, dt$$

$$= -W_t \, dt + (3W_t^2 - t)\, dW + \frac{1}{2} \cdot 6 W_t \, dt$$

$$= \underbrace{2W_t \, dt}_{\neq 0} + (3W_t^2 - t)\, dW$$

(Replace x with W_t in)

$\Rightarrow f(t, W_t)$ is NOT a mg.

Example 6.40. Let $X_t = t\sin(W_t)$. Is $X_t^2 - [X, X]_t$ a martingale?

Guess: IDK (X need not be a mg)

Ito: $f(t,x) = t \sin x$

$\partial_t f = \sin x$

$\partial_x f = t \cos x$

$\partial_x^2 f = -t \sin x$

$$dX_t = \partial_t f \, dt + \partial_x f \, dW + \frac{1}{2}\partial_x^2 f \, dt$$

$$= \sin(W_t)\,dt + t\cos W_t \, dW - \frac{1}{2} t \sin(W_t)\,dt$$

$$= \left(\left(1 - \frac{t}{2}\right) \sin W_t \right) dt + t \cos W_t \, dW.$$

$$\Rightarrow d[X,X]_t = t^2 \cos^2(W_t)\, dt$$

NTS Check $X^2 - [X,X]$ is a mg.

$$Y = X^2 - [X,X] \Rightarrow dY = 2X\,dX + \frac{1}{2} \cdot 2 \cdot d[X,X] - d[X,X]$$

$$= 2X\,dX$$

$$= 2X\left(\left(1-\frac{t}{2}\right)\sin W_t\right)dt \;\Rightarrow\; \neq 0$$

$$+ 2X\,t\cos W_t\;dW$$

$\Rightarrow X^2 - [X,X]$ is not a mg!

7. Review Problems

$W \to$ B.M.

Problem 7.1. If $0 \leqslant r \leqslant s \leqslant t$, find $\boldsymbol{E}(W_s W_t)$ and $\boldsymbol{E}(W_r W_s W_t)$.

Guess $E W_r^3 = 0$

$$E(W_s W_t) = s \wedge t \qquad (\min\{s, t\})$$
$$= s$$

(Sol 1: $E(W_s W_t) \overset{\text{tower}}{=} E\, E_s(W_s W_t) = E(W_s \underbrace{E_s W_t}_{\text{Mg}})$

$= E(W_s W_s) = s \qquad (W_s \sim N(0, s))$)

Sol 2: $\mathbb{E} W_s W_t = \mathbb{E} W_s (W_s + W_t - W_s)$

$$= \underbrace{\mathbb{E} W_s^2}_{} + \underbrace{\mathbb{E} W_s (W_t - W_s)}_{}$$

$$= s \qquad\qquad 0$$

($\because W_t - W_s$ is ind of W_s

$W_t - W_s \sim N(0, t-s)$)

Compute $\mathbb{E}(W_r W_s W_t)$ $\qquad r \leq s \leq t$

$$= E\left(W_r E_s\left(W_s W_t\right)\right)$$

$$= E\left(W_r W_s \; E_s W_t\right) = E\left(W_r W_s^2\right)$$

$$= E \, E_r\left(W_r W_s^2\right) = E\left(W_r \; E_r W_s^2\right)$$

$$= E\left(W_r \; E_r\left(W_s^2 - s + s\right)\right)$$

$$= E\left(W_r \left(W_r^2 - r + s \right) \right) \quad \left(\because W_s^2 - s \text{ is a mg} \right)$$

$$= E W_r^3 + E W_r (s - r) = 0$$

Problem 7.2. Define the processes X, Y, Z by

$$X_t = \int_0^{W_t} e^{-s^2}\,ds, \quad Y_t = \exp\left(\int_0^t W_s\,ds\right), \quad Z_t = tX_t^2$$

Decompose each of these processes as the sum of a martingale and a process of finite first variation. What is the quadratic variation of each of these processes?

① Write $X = X_0 + \underbrace{B}_{BV} + \underbrace{M}_{Mg}$

Usual strategy: $X_t = f(t, W_t)$ & apply Itô

$$dX = (\qquad)\,dt + (\qquad)\,dW$$

BV part Mg part.

Let $f(t, x) = \int_0^x e^{-s^2}\, ds \Rightarrow X_t = f(t, W_t)$

$\partial_t f = 0$

$\partial_x f = e^{-x^2}.$ (FTC)

$\partial_x^2 f = -2x\, e^{-x^2}$ (Chain rule)

d

$$dX_t = d f(t, X_t) = \partial_t f \, dt + \partial_x f \, dW + \frac{1}{2} \partial_x^2 f \, d\underbrace{[W, W]}_{dt}$$

$$= 0 \, dt + e^{-W_t^2} dW - \frac{1}{2} 2 W_t e^{-W_t^2} dt$$

$$= \left(- W_t e^{-W_t^2} dt \right) + \left(e^{-W_t^2} dW_t \right)$$

$$\Rightarrow X_t = X_0 + \underbrace{\int_0^t -W_s e^{-W_s^2} ds}_{B_t} + \int_0^t e^{-W_r^2} dW_r$$

$X_0 = 0$

$$B_t = -\int_0^t W_s e^{-W_s^2} ds$$

$$M_t = \int_0^t e^{-W_r^2} dW_r$$

$$Y_t = \text{exp}\left(\underbrace{\int_0^{t} W_s \, ds}\right)$$ ← already a diff fn of t (finite 1st var)

Note $g(t) = \int_0^t W_s \, ds$ → diff fn of t

$$Y_t = \underset{\sim}{Y_0} + \underbrace{(Y_t - Y_0)} + 0$$

finite 1st var (because Y is diff)

$Z_t = f(t, X_t) \quad , \quad f(t,x) = t\,x^2.$

& Just Ito to decompose Z

Problem 7.3. Define the processes X, Y by

$$X_t \stackrel{\text{def}}{=} \int_0^t W_s \, ds \,, \quad Y_t \stackrel{\text{def}}{=} \int_0^t W_s \, dW_s \,.$$

Given $0 \leqslant s < t$, compute $\boldsymbol{E} X_t$, $\boldsymbol{E} Y_t$, $\boxed{\boldsymbol{E}_s X_t}$, $\boldsymbol{E}_s Y_t$.

Notation

$$\left(E_s X_t = \underbrace{E\left(X_t \mid \mathcal{F}_s\right)}\right)$$

(1) $E \underbrace{\int_0^t W_s \, ds}_{\text{Riemann int}} = \int_0^t E W_s \, ds = 0$

$E \int dt = \int E \, dt$

(2) $$E_s \int_0^t W_r \, dr = \int_0^t E_s W_r \, dr$$

(Riemann Int)

$$= \int_0^s E_s W_r \, dr + \int_s^t E_s W_r \, dr$$

$$= \int_0^s W_r \, dr + \int_s^t W_s \, dr$$

$$= \int_0^s W_r\,dr + W_s(t-s)$$

② $Y_t = \int_0^t W_s\,dW_s$. Find EY_t & $E_s Y_t$

ⓐ $$E \underbrace{\int_0^t W_s\,dW_s}_{M_g} = \int_0^0 W_s\,dW_s = 0$$

(b) $E_s Y_t = E(Y_t \mid \mathcal{F}_s) = E_s \int_0^t W_r \, dW_r$

$$\underline{\underline{Mg}} \int_0^s W_s \, dW_s$$

Ito integrals are Mg's

QV of $\int_0^t \sigma_s \, dW_s$ is $\int_0^t \sigma_s^2 \, ds$

& $E\left(\int_0^t \sigma_s \, dW_s\right)^2 \overset{\text{Ito isom}}{=} E \int_0^t \sigma_s^2 \, ds$

Problem 7.4. Let $M_t = \int_0^t W_s \, dW_s$. Find a function f such that

$$\mathcal{E}(t) \stackrel{\text{def}}{=} \exp\left(M_t - \int_0^t f(s, W_s)\, ds\right)$$

is a martingale.

Let $g(t, x) = \exp\left(x - \int_0^t f(s, W_s)\, ds \right)$

$\mathcal{E}(t) = g(t, M_t)$

① $\partial_t g = \exp(\quad) \cdot (-f(t, W_t))$

② $\partial_x g = \exp(\quad)\, 1$

(3) $\partial_x^2 g = \exp(\quad) \cdot 1$

(4) $d[M,M]_t = W_t^2 \, dt$

$$\Rightarrow d\mathcal{E}(t) = \partial_t f \, dt + \partial_x f \, dM + \frac{1}{2}\partial_x^2 f \, d[M,M]$$

$$= \mathcal{E}(t)\left[-f(t,W_t)\right] dt + \mathcal{E}(t) W_t \, dW_t +$$

$$+ \frac{1}{2}\mathcal{E}(t) W_t^2 \, dt$$

$$= \mathcal{E}(t)\left[-f(t,W_t) + \frac{1}{2}W_t^2\right]dt + \mathcal{E}(t)\, W_t \, dW_t$$

Choose f so that the dt term vanishes

$$\boxed{\Rightarrow f(t,x) = \frac{x^2}{2}}$$

i.e $\mathcal{E}(t) = \exp\left(\int_0^t W_s \, dW_s - \frac{1}{2}\int_0^t W_s^2 \, ds\right)$

is a mg

Problem 7.5. Suppose $\sigma = \sigma_t$ is a deterministic (i.e. non-random) process, and M is a martingale such that $d[M, M]_t = \sigma_t^2 \, dt$. (Say $M_0 = 0$)

$$\cancel{X_t = \int_0^t \sigma_u \, dW_u} \, .$$

(1) Given $\lambda, s, t \in \mathbb{R}$ with $0 \leqslant s < t$ compute $\boldsymbol{E} e^{\lambda M_t}$ and $\boldsymbol{E}_s e^{\lambda M_t - M_s}$
(2) If $r \leqslant s$ compute $\boxed{\boldsymbol{E} \exp(\lambda M_r + \mu(M_t - M_s))}$.
(3) What is the joint distribution of $(M_r, M_t - M_s)$?
(4) *(Lévy's criterion)* If $\boxed{d[M, M]_t = dt}$, then show that M is a standard Brownian motion.

Compute $E\, e^{\lambda M_t}$ = (Mgf of M_t)

$$f(t,x) = e^{\lambda x} . \qquad \partial_t f = 0, \quad \partial_x f = \lambda e^{\lambda x}, \quad \partial_x^2 f = \lambda^2 e^{\lambda x}$$

$$d[M,M] = \sigma_t^2 \, dt$$

Let $\underline{\underline{\varphi(t)}} = \mathbb{E}\, e^{\lambda M_t}$

$$d\left(e^{\lambda M_t}\right) \overset{\text{Ito}}{=\!=} \partial_t f \, dt + \partial_x f \, dM + \frac{1}{2}\partial_x^2 f \, d[M,M]$$

$$= 0 + \lambda e^{\lambda M_t} dM + \frac{1}{2}\lambda^2 e^{\lambda M_t} \sigma_t^2 \, dt$$

$$\Rightarrow \quad e^{\lambda M_t} - \underbrace{e^{\lambda M_0}}_{1} = \lambda \int_0^t e^{\lambda M_s} \, dM_s + \frac{1}{2}\lambda^2 \int_0^t e^{\lambda M_s} \sigma_s^2 \, ds$$

$$\Rightarrow \quad E e^{\lambda M_t} - 1 = \lambda \underbrace{E \int_0^t e^{\lambda M_s} \, dM_s}_{\substack{M_g \\ E(\,)=0}} + \frac{1}{2}\lambda^2 E \int_0^t e^{\lambda M_s} \sigma_s^2 \, ds$$

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not random.

$$\varphi(t) = E\, e^{\lambda M_t}$$

$$\Rightarrow \varphi(t) = 1 + 0 + \frac{1}{2}\lambda^2 \int_0^t \varphi(s)\, \sigma_s^2 \, ds$$

$$\Rightarrow \varphi'(t) = \frac{\lambda^2}{2} \varphi(t)\, \sigma_t^2$$

$$\Rightarrow \frac{\varphi'}{\varphi} = \frac{\lambda^2}{2} \sigma_t^2 \Rightarrow \frac{d}{dt}\left(\ln \varphi\right) = \frac{\lambda^2}{2} \sigma_t$$

$$\Rightarrow \quad \underbrace{\ln \varphi(t) - \ln(\varphi(0))}_{\ln\left(\frac{\varphi(t)}{\varphi(0)}\right)} = \frac{\lambda^2}{2} \int_0^t \sigma_s^2 \, ds$$

$$\Rightarrow \quad \varphi(t) = \varphi(0) \exp\left(\frac{\lambda^2}{2} \int_0^t \sigma_s^2 \, ds \right)$$

$$\Rightarrow \underbrace{E\, e^{\lambda M_t}}_{\text{Mgf of } M_t} = 1 \cdot \underbrace{\exp\left(\frac{\lambda^2}{2}\int_0^t \sigma_s^2\, ds\right)}_{\text{Mgf of } N\left(0,\ \int_0^t \sigma_s^2\, ds\right)}$$

$$\Rightarrow M_t \sim N\left(0, \int_0^t \sigma_s^2\, ds\right)$$

Use the same trick to compute

$$E_s\, e^{\lambda(M_t - M_s)}$$

$$E_s\, e^{\lambda M_t} - e^{\lambda M_s} = \lambda E_s \int_s^t e^{\lambda M_r}\, dM_r + \frac{1}{2}\lambda^2 E_s \int_s^t e^{\lambda M_r} \sigma_r^2\, dr$$

Get $$E_s\, e^{\lambda(M_t - M_s)} = \exp\left(\frac{\lambda^2}{2} \int_s^t \sigma_r^2\, dr\right) \leftarrow \text{MGF of normal.}$$

(smells like $M_t - M_s$ shall be ind of $\mathcal{F}_s$).

Note: $\underline{X}$ & $\underline{Y}$ are ind $\iff$ $\underbrace{E\left(e^{\lambda X + \mu Y}\right)}_{\text{Joint MGF}} = \underbrace{Ee^{\lambda X}\ Ee^{\mu Y}}_{\text{product of MGF's}}$

(for all λ, μ).

Lets compute $\mathbb{E}\left(e^{\lambda M_r + \mu(M_t - M_s)}\right)$ $\quad (r \le s \le t)$

$$= \mathbb{E}\,\mathbb{E}_s\left(\qquad \right)$$

$$= \mathbb{E}\,\mathbb{E}_s\left(e^{\lambda M_r}\, e^{\mu(M_t - M_s)}\right) = \mathbb{E}\left(e^{\lambda M_r}\, \underbrace{\mathbb{E}_s\, e^{\mu(M_t - M_s)}}\right)$$

$$= E\left(e^{\lambda M_r} \cdot e^{\frac{\mu^2}{2}\int_s^t \sigma_u^2 \, du} \right)$$

$$= e^{\frac{\lambda^2}{2}\int_0^r \sigma_u^2 \, du + \frac{\mu^2}{2}\int_s^t \sigma_u^2 \, du}$$

$$= \text{MGF of a 2D normal mean } 0 \text{ \& Covariance } \begin{pmatrix} \int_0^r \sigma_u^2 \, du & 0 \\ 0 & \int_s^t \sigma_u^2 \, du \end{pmatrix}$$

$$\boxed{\Rightarrow M_t - M_s \text{ is ind of } M_r.}$$

If ~~Put~~ $\sigma = 1$ Get $M_t - M_s \sim N(0, t-s)$ & ind of $\mathcal{F}_s$ $\Big\}\Rightarrow M$ is a BM.

(Levy's Criterion)

Problem 7.6. Define the process X, Y by

$$\longrightarrow \quad X = \int_0^t s\,dW_s\,, \quad Y = \int_0^t W_s\,ds\,.$$

Find a formula for $\boldsymbol{E}X_t^n$ and $\boldsymbol{E}Y_t^n$ for any $n \in \mathbb{N}$.

Claim: Both X & Y are Normal!

not random.

$$X_t = \lim \underbrace{\sum t_i (W_{t_{i+1}} - W_{t_i})}_{\text{Linear combination of Normals!}} \implies X \text{ is Normal.}$$

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To find EX_t^n just find EX_t & EX_t^2

& use the formula for moments of Normal RV's.

$$EX_t = E\int_0^t s\,dW_s = 0$$

$$EX_t^2 = E\left(\int_0^t s\,dW_s\right)^2 \underset{=}{\overset{\text{Ito Isom}}{=}} E\int_0^t s^2\,ds = \frac{t^3}{3}.$$

4

Same for Y:

$$Y_t = \int_0^t W_s \, ds = \lim_{\|P\| \to 0} \sum \underbrace{W_{t_i}}_{\text{Normal}} \underbrace{(t_{i+1} - t_i)}_{\text{Not random}}$$

$$\underbrace{\sum W_{t_i} (t_{i+1} - t_i)}_{\text{Linear comb of Normal}}$$

$\Rightarrow Y$ is Normal.

(1) $$EY_t = E\int_0^t W_s\,ds = \int_0^t EW_s\,ds = 0$$

(2) $$EY_t^2 = E\left(\int_0^t W_s\,ds\right)^2 = E\left(\int_0^t W_s\,ds\right)\left(\int_0^t W_s\,ds\right)$$

$$= E\left(\int_0^t W_s\,ds \int_0^t W_r\,dr\right) = E\int_{s=0}^{t}\int_{r=0}^{t} W_s W_r\,ds\,dr$$

$$= \int_{s=0}^{t} \int_{r=0}^{t} E(W_s W_r) \, ds \, dr$$

$$= \int_{s=0}^{t} \int_{r=0}^{t} (s \wedge r) \, ds \, dr \quad \text{\& integrate!}$$

Get Mean & Variance & use MGF of Normal to find all moments!

8. Black Scholes Merton equation

- Cash: simple interest rate r in a bank.
- Let $\boxed{\Delta t}$ be small. $C_{n\,\Delta t}$ be cash in bank at time $n\,\Delta t$. $(n \in \mathbb{N})$
- Withdraw at time $n\,\Delta t$ and immediately re-deposit: $C_{(n+1)\Delta t} = (1 + r\,\Delta t)C_{n\Delta t}$.
- Set $t = n\Delta t$, send $\Delta t \to 0$: $\boxed{\partial_t C = rC}$ and $C_t = C_0 e^{rt}$.
- r is called the continuously compounded interest rate.
- Alternately: If a bank pays interest rate ρ after time T, then the equivalent continuously compounded interest rate is $r = \frac{1}{T}\ln(1+\rho)$.

$$C_{(n+1)\Delta t} - C_{n\Delta t} = r\,\Delta t\, C_{n\Delta t}$$

$$\Rightarrow \frac{C_{t+\Delta t} - C_t}{\Delta t} = r\,C_t \quad : \text{Send } \Delta t \to 0$$

$$\partial_t C_t$$

($\alpha \to$ Mean return rate)

- Stock price: $S_{t+\Delta t} = (1 + \overset{\alpha}{\cancel{r}}\,\Delta t) S_t + \boxed{\text{noise.}}$
  - ▷ Variance of noise should be proportional to Δt.
  - ▷ Variance of noise should be proportional to S_t.
- $S_{t+\Delta t} - S_t = \overset{\alpha}{\cancel{r}} S_t\, \Delta t + \sigma S_t (\Delta W_t)$.

Definition 8.1. A Geometric Brownian motion with parameters α, σ is defined by:

$$dS_t = \alpha S_t\, dt + \sigma S_t\, \boxed{dW_t}\,.$$

- α: Mean return rate (or percentage drift)
- σ: volatility (or percentage volatility)

Model for Stock price.

Proposition 8.2. $\boxed{S_t = S_0 \exp\left(\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)}$

Set $Y = \ln S_t$ & Itô:

$f(t,x) = \underline{\underline{\ln x}}$

$\partial_t f = 0 \quad \partial_x f = \frac{1}{x} \quad \partial_x^2 f = -\frac{1}{x^2}.$

$\Rightarrow dY = \partial_t f \, dt + \partial_x f \, dS + \frac{1}{2}\partial_x^2 \, d[S,S]$

$d[S,S] = \sigma^2 S_t^2 \, dt$

$= 0 + \frac{1}{S_t}(\alpha S dt + \sigma S dW) - \frac{1}{2S_t^2}\sigma^2 S_t^2 dt$

$$= \alpha\, dt + \sigma\, dW - \frac{\sigma^2}{2} dt$$

$$dY = \left(\alpha - \frac{\sigma^2}{2}\right) dt + \sigma\, dW \qquad (\alpha, \sigma \text{ const})$$

$$Y_t - Y_0 = \left(\alpha - \frac{\sigma^2}{2}\right) t + \sigma W_t$$

$$\ln\left(\frac{S_t}{S_0}\right) = \left(\alpha - \frac{\sigma^2}{2}\right) t + \sigma W_t \Rightarrow S_t = S_0 \exp\left(\left(\alpha - \frac{\sigma^2}{2}\right) t + \sigma W_t\right)$$

(Note $\ln x$ is not diff at $x = 0$, but Ito works because $S_t > 0 \;\forall t \geq 0$).

Market Assumptions.

$dS_t = \alpha S_t \, dt + \sigma S_t \, dW_t$

- 1 stock. Price S_t, modelled by GBM(α, σ).
- Money market: Continuously compounded interest rate r.
  - ▷ C_t = cash at time $t = C_0 e^{rt}$. (Or $\partial_t C_t = r C_t$.)
  - ▷ Borrowing and lending rate are both r.
- Frictionless (no transaction costs)
- Liquid (fractional quantities can be traded)

Consider a security that pays $V_T = g(S_T)$ at maturity time T.

Theorem 8.3. *If the security can be replicated, and $f = f(t, x)$ is a function such that the wealth of the replicating portfolio is given by $X_t = f(t, S_t)$, then:*

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$$\partial_t f + rx\partial_x f + \frac{\sigma^2 x^2}{2}\partial_x^2 f - rf = 0 \qquad x > 0,\ t < T\,, \tag{8.1}$$

(B.S.M PDE)

$$f(t, 0) = g(0)e^{-r(T-t)} \qquad t \leqslant T\,, \tag{8.2}$$

(Boundary condition)

$$f(T, x) = g(x) \qquad x \geqslant 0\,. \tag{8.3}$$

(Terminal condition)

Theorem 8.4. *Conversely, if f satisfies (8.1)–(8.3) then the security can be replicated, and $X_t = f(t, S_t)$ is the wealth of the replicating portfolio at any time $t \leqslant T$.*

Remark 8.5. Wealth of replicating portfolio equals the arbitrage free price.

Remark 8.6. $g(x) = (x - K)^+$ is a European call with strike K and maturity T.

Remark 8.7. $g(x) = (K - x)^+$ is a European put with strike K and maturity T.

Proposition 8.8. *A standard change of variables gives an explicit solution to* (8.1)–(8.3)*:*

$$f(t,x) = \int_{-\infty}^{\infty} e^{-r\tau} g\Big(x \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)\tau + \sigma\sqrt{\tau}\, y\Big)\Big) \frac{e^{-y^2/2}dy}{\sqrt{2\pi}}\,, \qquad \tau = T - t\,. \tag{8.4}$$

Corollary 8.9. *For European calls,* $g(x) = (x-K)^+$*, and*

$$f(t,x) = c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)) \tag{8.5}$$

where

$$d_\pm(\tau,x) \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big)\,, \tag{8.6}$$

$\tau = T - t$

and

$$N(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy\,, \tag{8.7}$$

$= P(N(0,1) < x)$

is the CDF of a standard normal variable.

Remark 8.10. Equation (8.1) is called a *partial differential equation.* In order to have a unique solution it needs:

PDE

(1) A terminal condition (this is equation (8.3)),
(2) A boundary condition at $x = 0$ (this is equation (8.2)),
(3) A boundary condition at infinity (not discussed yet).
  - ▷ For put options, $g(x) = (K - x)^+$, the boundary condition at infinity is

$$\lim_{x\to\infty} f(t,x) = 0\,.$$

  - ▷ For call options, $g(x) = (x - K)^+$, the boundary condition at infinity is

$$\lim_{x\to\infty} \left[f(t,x) - (x - Ke^{-r(T-t)})\right] = 0 \qquad \text{or} \qquad \boxed{f(t,x) \approx (x - Ke^{-r(T-t)}) \quad \text{as } x \to \infty\,.}$$

Expect S_t is $\gg K$, $S_T \gg K$ & payoff is $(S_T - K)$

forward contract.

Definition 8.11. If X_t is the wealth of a self-financing portfolio then

$$\boxed{dX_t = \Delta_t \, dS_t + r(X_t - \Delta_t S_t) \, dt}$$

for some adapted process Δ_t (called the trading strategy).

Disc time: Self fin $\longrightarrow$ no ext cash flows

$$X_{t+\delta t} = \Delta_t S_{t+\delta t} + (X_t - \Delta_t S_t)(1 + r\,\delta t)$$

Position at time t: Δ_t shares of stock

Rest cash.

$$\Rightarrow \quad X_{t+\delta t} - X_t = \Delta_t \left(S_{t+\delta t} - S_t \right) + \left(X_t - \Delta_t S_t \right) r \, \delta t$$

Send $\delta t \to 0$:

$$dX_t = \Delta_t \, dS_t + \left(X_t - \Delta_t S_t \right) r \, dt$$

Proof of Theorem 8.3. Assume X_t = wealth of Rep port

$= f(t, S_t)$

NTS f satisfies the BSM PDE

$X_t = f(t, S_t)$. Know $dX_t = \Delta_t \, dS_t + (X_t - \Delta_t S_t) r \, dt$

$$dS_t = \alpha S \, dt + \sigma S_t \, dW_t$$

$$\Rightarrow dX = \Delta_t (\alpha S dt + \sigma S dW) + r(X_t - \Delta_t S_t) dt$$

$$\boxed{(*) \quad dX_t = (r X_t + (\alpha - r)\Delta_t S_t)\, dt + \sigma \Delta_t S_t\, dW_t}$$

Also $X_t = f(t, S_t)$

$\Rightarrow$ Itô: $dX_t = \partial_t f \, dt + \partial_x f \, dS + \frac{1}{2} \partial_x^2 f \, d[S,S]$

$$= \partial_t f \, dt + \partial_x f \left(\alpha S \, dt + \sigma S \, dW \right) + \frac{1}{2} \partial_x^2 f \, \sigma^2 S^2 \, dt$$

$$\boxed{(**) \quad dX = \left(\partial_t f + \alpha S \partial_x f + \frac{1}{2} \sigma^2 S^2 \partial_x^2 f \right) dt + \sigma S \partial_x f \, \underline{dW}}$$

Uniqueness of S. Mg decomp $\Longrightarrow$ the dt terms & dW terms in (*) & (**) have to be equal.

$$\Rightarrow \sigma S \partial_x f = \sigma S \Delta_t \Rightarrow \partial_x f = \Delta_t$$

(Note $\partial_x f = \partial_x f(t, S_t)$)

Delta Hedging Rule $\longrightarrow$ $\boxed{\Rightarrow \partial_x f(t, S_t) = \Delta_t \quad (***)}$

Equate dt terms in $(*)$ & $(**)$:

Get $\quad \partial_t f + \alpha S \partial_x f + \frac{1}{2}\sigma^2 S^2 \partial_x^2 f = r X_t + (\alpha - r)\Delta_t S_t$

Know $X_t = f(t, S_t)$ & $\Delta_t = \partial_x f = \partial_x f(t, S_t)$

$$\Rightarrow \partial_t f + \underbrace{\alpha S \partial_x f} + \frac{1}{2}\sigma^2 S^2 \partial_x^2 f = r f + (\alpha - r) \underbrace{S \partial_x f}$$

$$\boxed{\Rightarrow \partial_t f + r \underset{x}{S} \partial_x f + \frac{\sigma^2}{2} S^2 \partial_x^2 f = r f}$$

Write x instead of S & get BSM PDE!!

Proof of Theorem 8.4.

Say f solves BS PDE

NTS $f(t, S_t) =$ wealth of R- port

Let $X_t =$ wealth of a self fin port with $X_0 = f(0, S_0)$

& $\Delta_t = \partial_x f(t, S_t)$

Set $Y_t = e^{-rt} X_t$

Know $dX_t = (rX_t + (\alpha - r)S_t)\,dt + \sigma \Delta_t S_t\, dW$

$$\boxed{\text{Choose } \Delta_t = \partial_x f(t, S_t) \quad (\text{Delta Hedging})}$$

Set $Y = e^{-rt} X_t$

$\Rightarrow$ By Itô, $dY = -r e^{-rt} X_t\, dt + e^{-rt} dX + 0$

$$\Rightarrow dY = -rY + e^{-rt}\left(rX_t + \Delta_t(\alpha - r)S_t\right)dt + e^{-rt}\sigma\,\Delta_t S_t\,dW$$

$$\boxed{\Delta_t = \partial_x f} \rightarrow \boxed{dY_t = e^{-rt}\Delta_t(\alpha - r)S_t\,dt + e^{-rt}\sigma\,\Delta_t S_t\,dW_t}$$

(2) Compute $d\left(e^{-rt} f(t, S_t)\right)$

$$= \left(e^{-rt}\partial_t f - r e^{-rt} f\right) dt + e^{-rt}\partial_x f \, dS + \frac{1}{2} e^{-rt} \partial_x^2 f \, d[S,S]$$

$$= \left(e^{-rt}\partial_t f - r e^{-rt} f\right) dt + e^{-rt}\partial_x f \left(\alpha S dt + \sigma S dW\right)$$

$$+ \frac{1}{2} e^{-rt} \partial_x^2 f \, \sigma^2 S^2 dt$$

$$= \left(e^{-rt}\partial_t f - r e^{-rt} f + e^{-rt}\partial_x f \cdot \underline{\alpha} S + \frac{1}{2}\sigma^2 S^2 e^{-rt}\partial_x^2 f\right) dt$$

$$+ e^{-rt} \partial_x f S_t \sigma \, dW$$

$$= e^{-rt} \underbrace{\partial_x f}_{} \cdot (\alpha - r) S_t \, dt + \underbrace{e^{-rt} \partial_x f \, \sigma S_t}_{} \, dW$$

$$= dY \quad \Rightarrow \quad d\left(e^{-rt} f(t, S_t)\right) = d\left(e^{-rt} X_t\right)$$

Choose $X_0 = f(0, S_0)$

$\Rightarrow$ ~~$\forall$~~ for all $t < T$,

$$e^{-rt} f(t, S_t) = e^{-rt} X_t$$

$$\boxed{\Rightarrow f(t, S_t) = X_t}$$

Last time: Market $\to \begin{cases} \text{M.M} \to (\text{interest rate } r) \\ \text{Stock} \to GBM(\alpha, \sigma): \; dS = \alpha S\,dt + \sigma S\,dW \end{cases}$ $\quad C_t = C_0 e^{rt} \; (\partial_t C = rC)$

Security with payoff $V_T = g(S_T)$ at time T

B.S.M. PDE: $\quad \partial_t f + r x \partial_x f + \frac{\sigma^2}{2} x^2 \partial_x^2 f = r f$

T.C.: $\quad f(T, x) = g(x)$

& B.C.

Last time : ① If $X_t = f(t, S_t)$ is the wealth of the rep portfolio, then f solves the B.S.M PDE (with BC & T.C. $f(T, x) = g(x)$)

② Conversely if f solves the BSM PDE (& B.C. & T.C) Then the security can be replicated & $X_t = f(t, S_t)$ is the wealth of the R. port.

Proof of Theorem 8.4. Last time:

Choose $X_0 = \underline{f(0, S_0)}$

Choose $\Delta_t = \underbrace{\partial_x f(t, S_t)}$

$\Bigg\}$ Let X_t = wealth of a self fin Port with initial capital $\underline{X_0}$ & holds Δ_t shares of stock at time t.

Set $Y_t = \underline{\underline{e}}^{-rt} X_t$. (Recall $dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t)dt$)

Compute dY_t = last time = $d\left(e^{-rt} f(t, S_t)\right)$

$$\Rightarrow d\left(Y_t - e^{-rt} f(t, S_t)\right) = 0$$

$$\Rightarrow Y_t - e^{-rt} f(t, S_t) - \left(Y_0 - f(0, S_0)\right) = \int_0^t 0 \, ds + \int_0^t 0 \, dW = 0$$

$$\Rightarrow e^{-rt} X_t - e^{-rt} f(t, S_t) = X_0 - f(0, S_0) = 0 \text{ (by choice of } X_0)$$

$\Rightarrow X_t = f(t, S_t)$

$\Rightarrow \underline{\underline{X_T}} = f(T, S_T) = g(S_T) = \underline{\underline{V_T}}$ = payoff of security

$\Rightarrow X$ = wealth of the Rep port.

$\Rightarrow f(t, S_t) =$

QED.

Proof of Theorem 8.4 (without discounting).

Start with ① $X_0 = f(0, S_0)$

② Choose $\Delta_t = \partial_x f(t, S_t)$ (Delta Hedging)

Want To Show: X is a rep port & $X_t = f(t, S_t)$

① By def of self fin: $dX_t = \Delta_t \, dS + r(X_t - \Delta_t S_t)\, dt$

$$\Rightarrow dX_t = \Delta_t (\alpha S \, dt + \sigma S \, dW) + r(X_t - \Delta_t S_t)\, dt$$

$$(*) \quad \Rightarrow dX_t = \sigma S \overset{\partial_x f(t,S_t)}{\Delta_t}\, dW_t + \left(rX_t + (\alpha - r)\Delta_t S_t\right) dt$$

② By Itô: $d\,\overbrace{f(t,S_t)}^{Y_t} = \partial_t f\, dt + \partial_x f\, dS + \frac{1}{2}\partial_x^2 f\, d[S,S]$

$$= \partial_t f\, dt + \partial_x f\left(\alpha S\, dt + \sigma S\, dW\right) + \frac{1}{2}\partial_x^2 f\, S^2\sigma^2 dt$$

$$\Rightarrow dY_t = \underbrace{\left(\partial_t f + \alpha S \partial_x f + \frac{\sigma^2 S^2}{2}\partial_x f\right)}_{4} dt + \underbrace{\partial_x f}_{\Delta_t}\, \sigma S\, dW$$

$$\boxed{\Rightarrow dY_t = \big((\alpha - r) S \partial_x f + r f\big) dt + \Delta_t \sigma S \, dW} \quad **$$

$$\left(\text{Using } \partial_t f + r x \partial_x f + \frac{\sigma^2}{2} x^2 \partial_x^2 f = r f \right)$$

$$\Rightarrow d(X_t - Y_t) = r\big(X_t - \underbrace{f(t, S_t)}_{Y}\big) dt + 0 \, dW$$

$$\Rightarrow d(X_t - Y_t) = r(X_t - Y_t) \, dt$$

$$\Rightarrow \quad \partial_t (X_t - Y_t) = r(X_t - Y_t)$$

$$\left.\begin{aligned} \Rightarrow X_t - Y_t &= (X_0 - Y_0) \cdot e^{rt} \\ &= \big(X_0 - f(0, S_0)\big) e^{rt} = 0 \end{aligned}\right\} \Rightarrow X_t = Y_t = f(t, S_t).$$

$$\Rightarrow X_T = Y_T = f(T, S_T) = g(S_T) = V_T$$

$\Rightarrow$ X is the wealth of the Rep Port.

& $X_t = f(t, S_t)$

Q.E.D.

Remark 8.12. The arbitrage free price does not $\boxed{\text{depend on the mean return rate!}}$

GBM: $\quad dS_t = \alpha S\, dt + \sigma S\, dW$

Mean return rate.

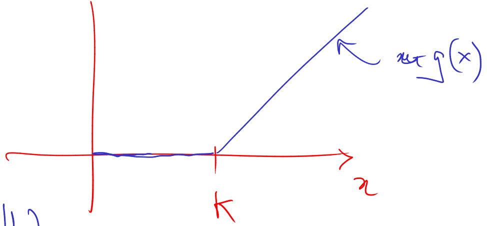
Question 8.13. *Consider a European call with maturity T and strike K. The payoff is $V_T = (S_T - K)^+$. Our proof shows that the arbitrage free price at time $t \leqslant T$ is given by $V_t = c(t, S_t)$, where c is defined by (8.5). The proof uses Itô's formula, which requires c to be twice differentiable in x; but this is clearly false at $t = T$. Is the proof still correct?*

$V_T = (S_T - K)^+ = g(S_T)$, where $g(x) = (x-K)^+$

Q: Is g diff (NO)

f solves BSM PDE

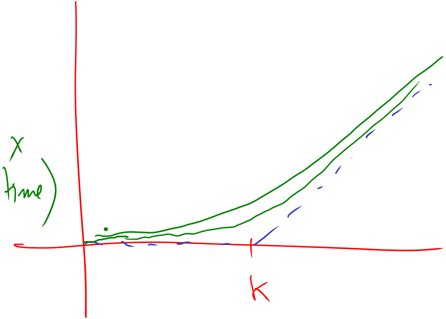
T.C. $f(T, x) = g(x) = (x-K)^+$ (not diff)



$f(t,x)$ for $t < T$

for $t < T$: f is diff (twice in x
one in time)

can apply Itô.



My proof will show $X_t = f(t, S_t)$ for all $t \underline{< T}$

take $\lim_{t \to T}$ & get $X_T = f(T, S_T)$.

Proposition 8.14 (Put call parity). *Consider a European put and European call with the same strike K and maturity T.*

- ▷ $c(t, S_t) =$ *AFP of call (given by (8.5))*
- ▷ $p(t, S_t) =$ *AFP of put.*

Then $c(t,x) - p(t,x) = x - Ke^{-r(T-t)}$, *and hence* $p(t,x) = Ke^{-r(T-t)} - x - c(t,x)$.

Know

$$c(t,x) = x\,N(d_+) - Ke^{-r(T-t)}N(d_-)$$

$$d_\pm = \frac{1}{\sigma\sqrt{t}}\left(\ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right)$$

8.3. **The Greeks.** Let $c(t,x)$ be the arbitrage free price of a European call with maturity T and strike K when the spot price is x. Recall

$$c(t,x) = xN(d_+) - Ke^{-r\tau}N(d_-)\,, \quad d_\pm \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big)\,, \quad \tau = T - t\,.$$

Definition 8.15. The *delta* is $\partial_x c$.

Remark 8.16 (Delta hedging rule). $\boxed{\Delta_t = \partial_x c(t, S_t).}$

Proposition 8.17. $\partial_x c = N(d_+)$

$$\partial_x c = \partial_x \left(x N(d_+) - K e^{-r\tau} N(d_-) \right)$$

$$= N(d_+) + x N'(d_+) \cdot d_+' - K e^{-r\tau} N'(d_-) d_-'$$

(1) $$d_\pm' = \partial_x \left(\frac{1}{\sigma\sqrt{\tau}} \left(\ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau \right) \right)$$

$$= \frac{1}{x\sigma\sqrt{\tau}}$$

(2) $$N(x) = \int_{-\infty}^{x} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}} \Rightarrow N'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

(3) $d_+^2 - d_-^2$:

$$d_\pm = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{x}{K}\right) + r\tau \pm \frac{\sigma^2\tau}{2}\right)$$

$$\Rightarrow d_+^2 - d_-^2 = 4\frac{1}{\sigma^2\tau}\left(\ln\left(\frac{x}{K}\right)+r\tau\right)\left(\frac{\sigma^2\tau}{2}\right)$$

$$= 2\left(\ln\left(\frac{x}{K}\right)+r\tau\right)$$

$$\Rightarrow e^{-d_-^2/2} = e^{-d_+^2/2 + \ln\left(\frac{x}{K}\right)+r\tau} = e^{-d_+^2/2}\,\frac{x}{K}\,e^{r\tau}$$

Hence $\partial_x c = N(d_+) + x N'(d_+) \cdot d'_+ - K e^{-r\tau} N'(d_-) d'_-$

$$= N(d_+) + d'_+ \left[x \frac{e^{-d_+^2/2}}{\sqrt{2\pi}} - K e^{-r\tau} \frac{e^{-d_-^2/2}}{\sqrt{2\pi}} \right]$$

$$= N(d_+) + \frac{d'}{\sqrt{2\pi}} \left(x e^{-d_+^2/2} - K e^{-r\tau} e^{-d_+^2/2} \frac{x}{K} e^{+r\tau} \right)$$

$$= N(d_+)$$

Definition 8.18. The $\boxed{\textit{Gamma} \text{ is}}$ $\partial_x^2 c$ and is given by $\partial_x^2 c = \dfrac{1}{x\sigma\sqrt{2\pi\tau}} \exp\left(\dfrac{-d_+^2}{2}\right)$.

Definition 8.19. The $\boxed{\textit{Theta}}$ is $\partial_t c$, and is given by $\partial_t c = -rKe^{-r\tau}N(d_-) - \dfrac{\sigma x}{2\sqrt{\tau}}N'(d_+)$

$$\partial_x^2 c = \partial_x \partial_x c = \partial_x \left(N(d_+) \right) = N'(d_+) \cdot d_+'$$

Proposition 8.20. (1) c *is* strictly *increasing as a function of* x.

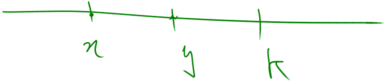
(2) c *is convex as a function of* x.

(3) c *is decreasing as a function of* t.

(1) c inc as a fn of x means $c(t, \underline{y}) > c(t, \underline{x})$ whenever $\underline{y} > \underline{x}$

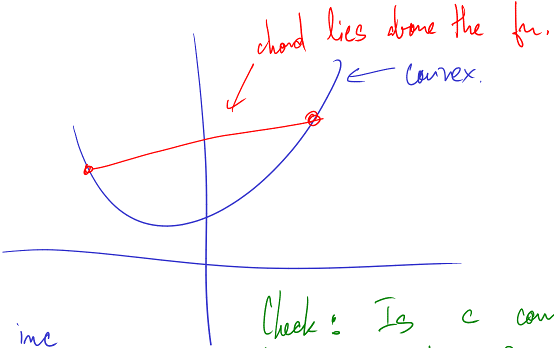
Pf: $\partial_x c > 0 \Rightarrow c$ is inc as a fn of x

$$\partial_x c = N(d_+) > 0$$



(2) Convex:

A fn is convex if the derivative is inc (i.e. the second derivative > 0)



Check: Is c convex as a fn of x?

i.e. Is $\partial_x^2 c > 0$?

$$\partial_x^2 c = \text{Gamma} = \frac{1}{x\sigma\sqrt{2\pi\tau}}\, e^{-d_+^2/2} > 0 \checkmark$$

---

(3) c is decreasing as a fn of time

(Check: $\partial_t c = \text{Theta} = \text{formula} < 0$)

Remark 8.21. To properly hedge a short call, you always borrow from the bank. Moreover $\Delta_T = 1$ if $S_T > K$, $\Delta_T = 0$ if $S_T < K$.

Delta Hedging: $\Delta_t =$ # shares in Rep port at time t

$$= \partial_x c(t, S_t)$$

$\Rightarrow$ Cash Balance: $c(t, S_t) - \partial_x c(t, S_t)\, S_t$ $\quad (\tau = T - t)$

Put $x = S_t$: $c(t,x) - x\,\partial_x c(t,x) = \cancel{xN(d_+)} - Ke^{-r\tau}N(d_-) - \cancel{xN(d_+)}$

$$= -Ke^{-r\tau}N(d_-) < 0$$

$$\underline{\underline{\Delta_t}} \xrightarrow{t \to T} \begin{cases} 1 & S_T > K \\ 0 & S_T < K \end{cases}$$

$$\Big(\text{Compute } \overbrace{\lim_{t \to T} \partial_x c(t, x)} = \begin{cases} 1 & x > K \\ 0 & x < K \\ 1/2 . & x = K \end{cases}$$

You check.

Remark 8.22 (Delta neutral, Long Gamma). Say x_0 is the spot price at time t.

- Short $\partial_x c(t, x_0)$ shares, and buy one call option valued at $c(t, x_0)$.
- Put $M = x_0 \partial_x c(t, x_0) - c(t, x_0)$ in the bank.
- What is the portfolio value when if the stock price is x (and we hold our position)?
  - ▷ *(Delta neutral)* Portfolio value $= c(t, x) -$ tangent line.
  - ▷ *(Long gamma)* By convexity, portfolio value is always non-negative.

x_0 = Spot price of stock

Portfolio - $\begin{cases} -\partial_x c(t, x_0) \text{ shares} \\ 1 \text{ call option.} \end{cases}$

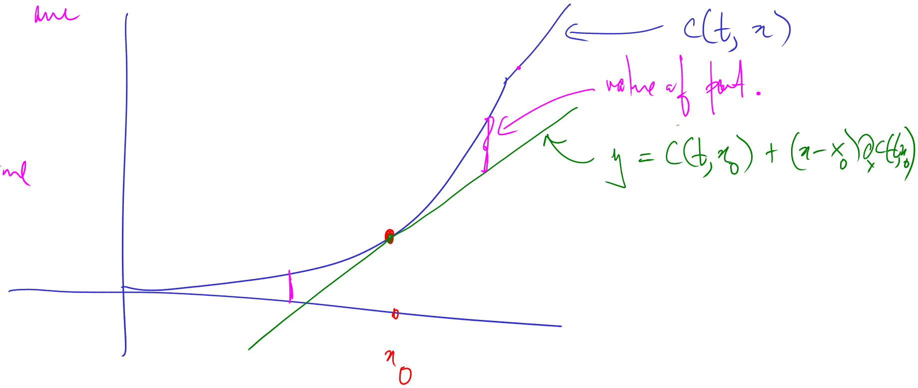
Portfolio value if spot price is x

$$= c(t,x) - x\,\partial_x c(t,x_0) + M$$

$$= c(t,x) - x\,\partial_x c(t,x_0) + x_0\,\partial_x c(t,x_0) - c(t,x_0)$$

$$= c(t,x) - \underbrace{\left[c(t,x_0) + (x - x_0)\,\partial_x c(t,x_0) \right]}_{\text{tangent line to } c(t,x) \text{ at } x_0}$$

Convex functions are always above the tangent line



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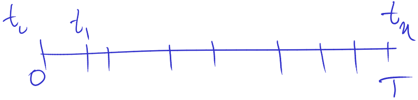
9. Multi-dimensional Itô calculus



- Let X and Y be two Itô processes.
- $P = \{0 = t_1 < t_1 \cdots < t_n = T\}$ is a partition of $[0, T]$.

$\|P\| = \max_i (t_{i+1} - t_i)$

Definition 9.1. The *joint quadratic variation* of X, Y, is defined by

$$[X, Y]_T = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}) \,,$$

$= \lim_{\|P\| \to 0} \sum_i (\Delta_i X)(\Delta_i Y)$

Remark 9.2. The joint quadratic variation is sometimes written as $d[X, Y]_t = dX_t \, dY_t$.

QV: $[X, X]_T = \lim_{\|P\| \to 0} \sum_{i=0}^{n} (\Delta_i X)^2$

$\Delta_i X = X_{t_{i+1}} - X_{t_i}$

Lemma 9.3. $\underbrace{[X,Y]_T}_{\text{Joint QV of } X \,\&\, Y} = \frac{1}{4}(\underbrace{[X+Y,X+Y]_T}_{\text{QV of } X+Y} - \underbrace{[X-Y,X-Y]_T}_{\text{QV of } X-Y})$

Pf: $(a+b)^2 - (a-b)^2 = 4ab$

$$\Rightarrow \underbrace{\lim_{\|P\|\to 0} \sum_i \Delta_i(\cancel{x+y}\,X)\,\Delta_i(\cancel{x-y}\,Y)}_{[X,Y]_T} = \lim \sum_i \underbrace{\frac{1}{4}\left[\left(\Delta_i(X+Y)\right)^2 - \left(\Delta_i(X-Y)\right)^2\right]}_{\frac{1}{4}\left([X+Y,X+Y]_T - [X-Y,X-Y]_T\right)}$$

Proposition 9.4 (Product rule). $d(XY)_t = X_t\, dY_t + Y_t dX_t + \boxed{d[X, Y]_t}$

If X & Y are diff,

$$\frac{d}{dt}(XY) = X\frac{dY}{dt} + \frac{dX}{dt}Y$$

$$\Rightarrow d(XY) = X\,dY + Y\,dY$$

If X, Y are stoch processes (not diff)

$$d(XY) = X\,dY + Y\,dX + d[X, Y]$$

Pf: $4XY = (X+Y)^2 - (X-Y)^2$

Itô: $d\left((X+Y)_t^2\right) = 2(X_t+Y_t)\, d(X_t+Y_t) + \frac{1}{2}\cdot 2\, d[X+Y, X+Y]$

$d\left((X-Y)_t^2\right) = 2(X_t-Y_t)\, d(X_t-Y_t) + d[X-Y, X-Y]_t$

$= 2X_t dX_t + 2Y_t dY_t - 2Y_t dX_t - 2X_t dY_t$

$+ d[X-Y, X-Y]_t.$

$$\Rightarrow d\left((X+Y)^2 - (X-Y)^2\right)$$

$$= 4X_t\, dY_t + 4Y_t\, dX_t + d\left([X+Y, X+Y]_t - [X-Y, X-Y]_t\right)$$

$$\Rightarrow 4\, d(XY) = 4X\, dY + 4Y\, dX + 4\, d[X, Y] \Rightarrow \text{done!}$$

Proposition 9.5. *Say X, Y are two semi-martingales.*

- *Write $X = X_0 + B + M$, where B has bounded variation and M is a martingale.*
- *Write $Y = Y_0 + C + N$, where C has bounded variation and N is a martingale.*
- *Then $d[X, Y]_t = d[M, N]_t$.*

Remark 9.6. Recall, all processes are implicitly assumed to be *adapted* and *continuous*.

Pf: $[X,Y] = \frac{1}{4}\Big([X+Y, X+Y] - [X-Y, X-Y]\Big)$

$= \frac{1}{4}\Big([M+N, M+N] - [M-N, M-N]\Big)$ ($\because$ BV part does not change QV)

$$= [M, N].$$

Corollary 9.7. *If X is a semi-martingale and B has bounded variation then $[X, B] = 0$.*

$[X, B] = [\text{mg part of } X, \text{ mg part of } B]$

$= [M, 0] = 0$

Notation.

- *d-dimensional vectors:* Write $x = (x_1, \dots, x_d) \in \mathbb{R}^d$.
- *d-dimensional* $\boxed{\text{random } \textit{vectors:}}$ $X = (X_1, \dots, X_d)$, where each X_i is a random variable.
- *d-dimensional stochastic processes:* $X_t = (X_t^1, \dots, X_t^d)$, where each X_t^i is a stochastic process.
  - ▷ For scalars (or random variables): X^i denotes the i-th power of X.
  - ▷ For vectors (or random random vectors): X^i denotes the i-th *coordinate* of X.
  - ▷ There is no ambiguity (can't take powers of vectors, or coordinates of scalars)
- Alternate notation used in many books: Use $X(t)$ for the d-dimensional stochastic process, and $X_i(t)$ for the i-th coordinate.
- May sometimes write $X = (X^1, \dots, X^d)$ for random vectors, instead of $(X_1, \dots, X_d)$.

Notation for d-dim Processes

X^{i}_{t} $i \longrightarrow$ coordinate

$t \longrightarrow$ time

Remark 9.8 (Chain rule). If X is a differentiable function of t, then

$$d(f(t, X_t)) = \partial_t f(t, X_t)\, dt + \sum_{i=1}^{d} \partial_i f(t, X_t)\, dX_t^i$$

Remark 9.9 (Notation). $\partial_t f = \frac{\partial f}{\partial t}$, $\partial_i f = \frac{\partial f}{\partial x_i}$.

$f = f(t, x)$, $x \in \mathbb{R}^d$

$$\frac{d}{dt}\Big(f(t, X_t)\Big) \overset{\text{Chain Rule}}{=} \partial_t f \Big|_{(t, X_t)} \frac{dt}{dt} + \sum_{i=1}^{d} \partial_i f \Big|_{(t, X_t)} \frac{dX^i}{dt}$$

$$\Rightarrow \frac{d}{dt} f(t, X_t) = \partial_t f(t, X_t) + \sum_{i=1}^{d} \partial_i f(t, X_t) \frac{dX^i}{dt}$$

"Multiply by dt" & get

$$\underbrace{df(t, X_t) = \partial_t f(t, X_t)\, dt + \sum_{i=1}^{d} \partial_i f(t, X_t)\, dX^i}$$

Theorem 9.10 (Multi-dimensional Itô formula).

- *Let* X *be a* d*-dimensional Itô process.* $X_t = (X_t^1, \dots, X_t^d)$.
- *Let* $f = f(t, x)$ *be a function that's defined for* $t \in \mathbb{R}$, $x \in \mathbb{R}^d$.
- *Suppose* $f \in C^{1,2}$. *That is:*
  - ▷ f *is once differentiable in* t
  - ▷ f *is twice in each coordinate* x_i (includes $\partial_i \partial_j f$)
  - ▷ *All the above partial derivatives are continuous. Then:*

extra!

$$\underbrace{d(f(t, X_t)) = \partial_t f(t, X_t)\, dt + \sum_{i=1}^{d} \partial_i f(t, X_t)\, dX_t^i}_{\text{Chain Rule}} + \overbrace{\frac{1}{2} \sum_{i,j} \partial_i \partial_j f(t, X_t)\, d[X^i, X^j]_t}^{\text{extra!}}$$

Joint QV.

Remark 9.11 (Integral form of Itô's formula).

$$f(T, X_T) - f(0, X_0) = \int_0^T \partial_t f(t, X_t)\, dt + \sum_{i=1}^{d} \int_0^T \partial_i f(t, X_t)\, dX_t^i$$
$$+ \frac{1}{2} \sum_{i,j} \int_0^T \partial_i \partial_j f(t, X_t)\, d[X^i, X^j]_t$$

Remark 9.12. As with the 1D Itô, will drop the arguments (t, X_t). *Remember they are there.*

$$\int_0^T \partial_t f\, dt + \sum_{i=1}^{d} \int \partial_i f\, dX_t^i + \frac{1}{2} \sum_{i,j} \int_0^T \partial_i \partial_j f\, d[X^i, X^j]_t$$

Intuition behind Theorem 9.10.

$$P = \{0 = t_0 < t_1 \dots t_n = T\}$$

$$f(T, X_T) - f(0, X_0) = \sum_{k=1}^{n} f(t_{k+1}, X_{t_{k+1}}) - f(t_k, X_{t_k})$$

$$\overset{\text{Taylor}}{=} \sum_{k=0}^{n-1} \underbrace{\partial_t f \, (t_{k+1} - t_k)}_{\xrightarrow{\|P\| \to 0} \int_0^T \partial_t f \, dt} + \sum_{k=0}^{n-1} \sum_{i=1}^{d} \underbrace{\partial_i f \, (X^i_{t_{k+1}} - X^i_{t_k})}_{\xrightarrow{\|P\| \to 0} \int_0^T \partial_i f \, dX^i}$$

$$+\sum_{k=0}^{n-1} \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \partial_i \partial_j f \underbrace{\left(X^i_{t_{k+1}} - X^i_{t_k}\right)\left(X^j_{t_{k+1}} - X^j_{t_k}\right)}_{\text{inc of Joint QV.}}$$

$$\xrightarrow{\|P\| \to 0} \frac{1}{2} \int_0^T \sum_{i=1}^{d} \sum_{j=1}^{d} \partial_i \partial_j f \, d\left[X^i, X^j\right]_t$$

To use the d-dimensional Itô formula, we need to compute joint quadratic variations.

Proposition 9.13. *Let M, N be continuous martingales, with $\boldsymbol{E} M_t^2 < \infty$ and $\boldsymbol{E} N_t^2 < \infty$.*

(1) $MN - [M, N]$ is also a continuous martingale.

(2) Conversely if $MN - B$ is a continuous martingale for some continuous adapted, bounded variation *process B with $B_0 = 0$, then $\boxed{B = [M, N]}$.*

Proof. 1 $d\big(MN - [M,N]\big) = M\,dN + N\,dM + \cancel{d[M,N]} - \cancel{d[M,N]}$

$$= \underbrace{M\,dN}_{Mg} + \underbrace{N\,dM}_{Mg}$$

(Recall: If M is a mg then $M^2 - [M, M]$ is ~~dso~~ also a mg)

Proposition 9.14. *(1)* (Symmetry) $[X, Y] = [Y, X]$

(2) (Bi-linearity) *If* $\alpha \in \mathbb{R}$, X, Y, Z *are semi-martingales,* $[X, Y + \alpha Z] = [X, Y] + \alpha[X, Z]$.

Proof.

Joint QV

$$[X, Y + \alpha Z]_T = \lim_{\|P\| \to 0} \sum (\Delta_i X)(\Delta_i (Y + \alpha Z))$$

$$= \lim_{\|P\| \to 0} \sum \Delta_i X \Delta_i Y + \alpha \sum (\Delta_i X)(\Delta_i Z)$$

$$= [X, Y]_T + \alpha [X, Z]_T$$

//

Proposition 9.15. *Let M, N be two martingales, σ, τ two adapted processes.*

- *Let* $X_t = \int_0^t \sigma_s \, dM_s$ *and* $Y_t = \int_0^t \tau_s \, dN_s$.
- *Then* $\cancel{[X, Y, X, Y]_t = \int_0^t \sigma_s \tau_s \, d[M, N]_s}$.

$$[X, Y]_t = \int_0^t \sigma_s \tau_s \, d[M, N].$$

Remark 9.16. Alternately, if $\boxed{dX_t = \sigma_t \, dM_t}$ and $\boxed{dY_t = \tau_t \, dN_t}$, then $d[X, Y]_t = \sigma_t \tau_t \, d[M, N]_t$.

Intuition.

Recall $[X, X]_t = \int_0^t \tau_s^2 \, d[M, M]_s$

$$X_T = \lim_{\|P\| \to 0} \sum \tau_{t_i} \Delta_i M \qquad \Delta_i X = X_{t_{i+1}} - X_{t_i}$$

?

$$Y_T = \lim_{\|P\| \to 0} \sum \tau_{t_i} \Delta_i N \quad \approx \sigma_{t_i} \left(M_{t_{i+1}} - M_{t_i} \right)$$

$$\Delta_i Y \approx \tau_{t_i} \left(N_{t_{i+1}} - N_{t_i} \right)$$

$$[X, Y] \approx \sum (\Delta_i X)(\Delta_i Y)$$

$$= \sum (\sigma_i \Delta_i M)(\tau_i \Delta_i N) \longrightarrow \int_0^T \sigma_t \tau_t \, d[M, N]_t$$

Proposition 9.17. *If M, N are continuous martingales, $\boldsymbol{E} M_t^2 < \infty$, $\boldsymbol{E} N_t^2 < \infty$ and M, N are independent, then $[M, N] = 0$.*

Remark 9.18 (Warning). Independence implies $\boldsymbol{E}(M_t N_t) = \boldsymbol{E} M_t \boldsymbol{E} N_t$. But it *does not* imply $\boldsymbol{E}_s(M_t N_t) = \boldsymbol{E}_s M_t \boldsymbol{E}_s N_t$. So you can't use this to show MN is a martingale, and hence conclude $[M, N] = 0$.

Correct proof.

$$\boxed{E_s(M_t N_t) \overset{\text{indep}}{=} E_s M_t \, E_s N_t} \leftarrow \text{FALSE}$$

$$= M_s N_s$$

$\Rightarrow M_t N_t$ is a mg $\Rightarrow [M, N] = 0$

WRONG (Not fixable!).

Correct Pf: Claim $E[M,N]_T^2 = 0$

$$E[M,N]_T^2 \approx E\left(\sum (\Delta_i M)(\Delta_i N)\right)^2$$

$$= E\left[\sum_{i,j} (\Delta_i M)(\Delta_i N)\ (\Delta_j M)(\Delta_j N)\right]$$

$$= E\sum_{i=0}^{n-1} (\Delta_i M)^2 (\Delta_i N)^2 + 2E\sum_{j=1}^{n-1} \sum_{i=0}^{j-1} (\Delta_i M)(\Delta_j M)(\Delta_i N)(\Delta_j N)$$

$$= \quad || \quad + 2 \sum_{j=1}^{n-1} \sum_{i=0}^{j-1} E(\Delta_i M \, \Delta_j M) \, E((\Delta_i N) \, \Delta_j N)$$

$$= \quad || \quad + 2 \sum \sum E \, \underbrace{E_{t_{i+1}} (\Delta_i M \, \Delta_j M)}_{0\,!} \cdot E(\quad)$$

$$\because E_{t_{i+1}}(\Delta_j M) = E_{t_{i+1}}(M_{t_{j+1}} - M_{t_j}) = M_{t_{i+1}} - M_{t_{i+1}} = 0$$

$$= E \sum_{i=0}^{n-1} (\Delta_i M)^2 (\Delta_i N)^2 + 0$$

$$= \sum_{i=0}^{n-1} E(\Delta_i M)^2 \; E(\Delta_i N)^2.$$

$$\leq \underbrace{\max_i E(\Delta_i M)^2}_{\downarrow \text{ (since } M \text{ is cts!)} \atop 0} \cdot \underbrace{\sum_i E(\Delta_i N)^2}_{\downarrow \atop E[N,N]_T \, < \, \infty} = 0$$

Remark 9.19. $[M, N] = 0$ does not imply M, N are independent. For example:

- ~~Let $M_t = \int_0^t \mathbf{1} W_s < 0 \, dW_s$~~
- ~~Let $N_t = \int_0^t \mathbf{1} W_s \geqslant 0 \, dW_s$~~

$$\left.\begin{aligned} M_t &= \int_0^t \mathbb{1}_{\{W_s < 0\}} \, dW_s \\ N_t &= \int_0^t \mathbb{1}_{\{W_s \geqslant 0\}} \, dW_s \end{aligned}\right\} \begin{aligned} d[M,N] &= \mathbb{1}_{\{W_s<0\}} \mathbb{1}_{\{W_s \geqslant 0\}} \, ds \\ &= 0 \, ds \end{aligned}$$

But $M_t + N_t = \int_0^t 1 \, dW_s = W_t \leftarrow \Rightarrow M$ & N are Not ind.

Definition 9.20 (d-dimensional Brownian motion)**.** We say a d-dimensional process $W = (W^1, \dots, W^d)$ is a Brownian motion if:

(1) Each coordinate W^i is a standard 1-dimensional Brownian motion.
(2) For $i \neq j$, the processes W^i and W^j are independent.

Remark 9.21. If W is a d-dimensional Brownian motion then $\boxed{d[W^i, W^j]_t = \begin{cases} dt & i = j\,, \\ 0\,dt & i \neq j\,. \end{cases}}$

Theorem 9.22 (Lévy)**.** *Let M be a d-dimensional process such that:*

(1) M is a continuous martingale.

(2) The joint quadratic variation satisfies: $d[W^i, W^j]_t = \begin{cases} dt & i = j\,, \\ 0\,dt & i \neq j\,. \end{cases}$

Then M is a d-dimensional Brownian motion.

Proof. Find $\boldsymbol{E}_s e^{\lambda M_t^i + \mu M_t^j}$ using Itô's formula, similar to Problem 7.5. $\square$

Example 9.23. Let $f \in C^{1,2}$, W be a d-dimensional Brownian motion, and set $X_t = f(t, W_t)$. Find the Itô decomposition of X.

Question 9.24. *Let W be a 2-dimensional Brownian motion. Let $X_t = \ln(|W_t|^2) = \ln((W_t^1)^2 + (W_t^2)^2)$. Is X a martingale?*

10. **Risk Neutral Pricing**

Goal.

- Consider a market with a bank and one stock.
- The interest rate R_t is some adapted process.
- The stock price satisfies $dS_t = \alpha_t S_t \, dt + \sigma_t S_t \, dW_t$. (Here α, σ are adapted processes).
- Find the risk neutral measure and use it to price securities.

Definition 10.1. Let $D_t = \exp(-\int_0^t R_s \, ds)$ be the discount factor.

Remark 10.2. Note $\partial_t D = -R_t D_t$.

Remark 10.3. D_t dollars in the bank at time 0 becomes \$1 in the bank at time t.

Theorem 10.4. *The (unique) risk neutral measure is given by* $d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}$*, where*

$$Z_T = \exp\Big(- \int_0^T \theta_t \, dW_t - \frac{1}{2} \int_0^T \theta_t^2 \, dt \Big) \,, \qquad \theta_t = \frac{\alpha_t - R_t}{\sigma_t} \,.$$

Theorem 10.5. *Any security can be replicated. If a security pays* V_T *at time* T*, then the arbitrage free price at time* t *is*

$$V_t = \frac{1}{D_t} \tilde{\boldsymbol{E}}_t(D_T V_T) \,.$$

Remark 10.6. We will explain the notation $d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}$ and prove both the above theorems later.

Definition 10.7. We say $\tilde{\boldsymbol{P}}$ is a risk neutral measure if:

(1) $\tilde{\boldsymbol{P}}$ is equivalent to $\boldsymbol{P}$ (i.e. $\tilde{\boldsymbol{P}}(A) = 0$ if and only if $\boldsymbol{P}(A) = 0$)
(2) $D_t S_t$ is a $\tilde{\boldsymbol{P}}$ martingale.

Remark 10.8. As before, if $\tilde{\boldsymbol{P}}$ is a new measure, we use $\tilde{\boldsymbol{E}}$ to denote expectations with respect to $\tilde{\boldsymbol{P}}$ and $\tilde{\boldsymbol{E}}_t$ to denote conditional expectations.

Example 10.9. Fix $T > 0$. Let Z_T be a $\mathcal{F}_T$-measurable random variable.

- Assume $Z_T > 0$ and $\boldsymbol{E} Z_T = 1$.
- Define $\tilde{\boldsymbol{P}}(A) = \boldsymbol{E}(Z_T \mathbf{1}_A) = \displaystyle\int_A Z_T \, d\boldsymbol{P}$.
- Can check $\tilde{\boldsymbol{E}} X = \boldsymbol{E}(Z_T X)$. That is $\displaystyle\int_\Omega X \, d\tilde{\boldsymbol{P}} = \int_\Omega X \, Z_T \, d\boldsymbol{P}$.
- Notation: Write $d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}$.

Lemma 10.10. *Let $Z_t = \boldsymbol{E}_t Z_T$. If X_t is $\mathcal{F}_t$-measurable, then $\tilde{\boldsymbol{E}}_s X = \frac{1}{Z_s} \tilde{\boldsymbol{E}}_s (Z_t X_t)$.*

Proof. You will see this in the proof of the Girsanov theorem in part 2 of this course. $\square$

Theorem 10.11 (Cameron, Martin, Girsanov)**.** *Fix $T > 0$, and define:*

- $b_t = (b_t^1, \dots, b_t^d)$ *a d-dimensional adapted process.*
- W *a d-dimensional Brownian motion.*
- $\tilde{W}_t = W_t + \int_0^t b_s \, ds$ *(i.e. $d\tilde{W}_t = b_t \, dt + d\tilde{W}_t$).*
- $d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}$, *where*

$$Z_t = \exp\Big(- \int_0^t b_s \cdot dW_s - \frac{1}{2} \int_0^t |b_s|^2 \, ds \Big) \, .$$

If Z is a martingale, then $\tilde{\boldsymbol{P}}$ is an equivalent measure under which $\tilde{W}$ is a Brownian motion up to time T.

Last time: Multi dim Ito

$$d f(t, X_t) = \partial_t f \, dt + \sum_{i=1}^{d} \partial_i f \, dX^i_t + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \partial_i \partial_j f \; \underbrace{d[X^i, X^j]_t}_{\text{Joint QV}}$$

Definition 9.20 (d-dimensional Brownian motion)**.** We say a d-dimensional process $W = (W^1, \dots, W^d)$ is a Brownian motion if:

(1) Each coordinate W^i is a standard 1-dimensional Brownian motion.
(2) For $i \neq j$, the processes W^i and W^j are independent.

Remark 9.21. If W is a d-dimensional Brownian motion then $\boxed{d[W^i, W^j]_t = \begin{cases} dt & i = j\,, \\ 0\,dt & i \neq j\,. \end{cases}}$

Joint QV
of ind cts mg $= 0$

Theorem 9.22 (Lévy). *Let M be a d-dimensional process such that:*

(1) *M is a continuous martingale.*

(2) *The joint quadratic variation satisfies:* d[M^i, M^j]

$$d[\cancel{W}^i, \cancel{W}^j]_t = \begin{cases} dt & i = j\,, \\ 0\,dt & i \neq j\,. \end{cases}$$

Then M is a d-dimensional Brownian motion.

Proof. Find $\boxed{\boldsymbol{E}_s e^{\lambda M_t^i + \mu M_t^j}}$ using Itô's formula, similar to Problem 7.5. (Recitation/Review) □

$\Rightarrow$ M^i & M^j are ind for $i \neq j$

& $\Rightarrow$ $M_t - M_s \sim N\left(0, \underbrace{\begin{pmatrix} t-s & & \\ & t-s & \\ & & \ddots & \\ & & & t-s \end{pmatrix}}_{(t-s)I}\right)$

Example 9.23. Let $f \in C^{1,2}$, W be a d-dimensional Brownian motion, and set $X_t = f(t, W_t)$. Find the Itô decomposition of X.

(Maybe later)

Question 9.24. *Let W be a 2-dimensional Brownian motion. Let $X_t = \ln(|W_t|^2) = \ln((W_t^1)^2 + (W_t^2)^2)$. Is X a martingale?*

(later)

10. Risk Neutral Pricing

$\iff C_t = C_0 \exp\left(\int_0^t R_s \, ds\right)$

Cash in bank $\partial_t C_t = R_t \cdot C_t$

Goal.

- Consider a market with a bank and one stock.
- The interest rate R_t is some adapted process.
- The stock price satisfies $dS_t = \alpha_t S_t \, dt + \sigma_t S_t \, dW_t$. (Here α, σ are adapted processes).
- Find the risk neutral measure and use it to price securities.

Definition 10.1. Let $D_t = \exp(-\int_0^t R_s \, ds)$ be the discount factor.

Remark 10.2. Note $\partial_t D = -R_t D_t$.

Remark 10.3. D_t dollars in the bank at time 0 becomes \$1 in the bank at time t.

T = maturity time

Theorem 10.4. *The (unique) risk neutral measure is given by* $\boxed{d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}}$, *where*

$$Z_T = \exp\Big(-\int_0^T \theta_t \, dW_t - \frac{1}{2}\int_0^T \theta_t^2 \, dt \Big) \,, \qquad \boxed{\theta_t = \frac{\alpha_t - R_t}{\sigma_t}} \,.$$

Theorem 10.5. *Any security can be replicated. If a security pays* V_T *at time* T, *then the arbitrage free price at time* t *is*

$$V_t = \frac{1}{D_t} \tilde{\boldsymbol{E}}_t (D_T V_T) = \tilde{\boldsymbol{E}}_t \Big(\exp\Big(\int_t^T -R_s \, ds \Big) \, V_T \Big) \Big) \,.$$

Remark 10.6. We will explain the notation $d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}$ and prove both the above theorems later.

Same formula as in the Binomial model.

Definition 10.7. We say $\tilde{\boldsymbol{P}}$ is a risk neutral measure if:

(1) $\tilde{\boldsymbol{P}}$ is equivalent to $\boldsymbol{P}$ (i.e. $\tilde{\boldsymbol{P}}(A) = 0$ if and only if $\boldsymbol{P}(A) = 0$)

(2) $D_t S_t$ is a $\tilde{\boldsymbol{P}}$ martingale.

Remark 10.8. As before, if $\tilde{\boldsymbol{P}}$ is a new measure, we use $\tilde{\boldsymbol{E}}$ to denote expectations with respect to $\tilde{\boldsymbol{P}}$ and $\tilde{\boldsymbol{E}}_t$ to denote conditional expectations.

Example 10.9. Fix $T > 0$. Let Z_T be a $\mathcal{F}_T$-measurable random variable.

- Assume $Z_T > 0$ and $\boldsymbol{E} Z_T = 1$. $(Z > 0 \Rightarrow P(A) > 0 \iff \tilde{P}(A) > 0)$
- Define $\tilde{\boldsymbol{P}}(A) = \boldsymbol{E}(Z_T \mathbf{1}_A) = \int_A Z_T \, d\boldsymbol{P}$. $(E Z_T = 1 \Rightarrow \tilde{P}(\Omega) = E Z_T 1_\Omega = E Z_T = 1)$
- Can check $\tilde{\boldsymbol{E}} X = \boldsymbol{E}(Z_T X)$. That is $\int_\Omega X \, d\tilde{\boldsymbol{P}} = \int_\Omega X \, Z_T \, d\boldsymbol{P}$.
- Notation: Write $\boxed{d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}}$.

Lemma 10.10. *Let $Z_t = \boldsymbol{E}_t Z_T$. If X_t is $\mathcal{F}_t$-measurable, then* $\tilde{\boldsymbol{E}}_s X_t = \frac{1}{Z_s} \boldsymbol{E}_s(Z_t X_t)$.

Proof. You will see this in the proof of the Girsanov theorem. $\square$

Theorem 10.11 (Cameron, Martin, Girsanov). *Fix $T > 0$, and define:*

- $b_t = (b_t^1, \dots, b_t^d)$ *a d-dimensional adapted process.*
- W *a d-dimensional Brownian motion.*
- $\tilde{W}_t = W_t + \int_0^t b_s\, ds$ *(i.e. $d\tilde{W}_t = b_t\, dt + dW_t$).*
- $d\tilde{\boldsymbol{P}} = Z_T\, d\boldsymbol{P}$*, where*

$(B_t^i = \int_0^t b_s^i\, ds$

$$Z_t = \exp\Big(-\int_0^t b_s \cdot dW_s - \frac{1}{2}\int_0^t |b_s|^2\, ds \Big)\,.$$

If Z is a martingale, then $\tilde{\boldsymbol{P}}$ is an equivalent measure under which $\tilde{W}$ is a Brownian motion up to time T.

Remark 10.12. Note $\tilde{W}_t$ is a vector.

(1) So $\tilde{W}_t = W_t + \int_0^t b_s\, ds$ means $\tilde{W}_t^i = W_t^i + \int_0^t b_s^i\, ds$, for each $i \in \{1, \dots, d\}$.
(2) Similarly, $d\tilde{W}_t = b_t\, dt + dW_t$ means $d\tilde{W}_t^i = b_t^i\, dt + dW_t^i$ for each $i \in \{1, \dots, d\}$.

Remark 10.13. $\int_0^t b_s \cdot dW_s$ means $\int_0^t \sum_{i=1}^d b_s^i\, dW_s^i$ (dot product).

(Sometimes $E\int_0^t Z_s|b_s|^2 ds = \infty$)

Proposition 10.14. $dZ_t = \boxed{-Z_t b_t \cdot dW_t}$. *Explicitly, in coordinates,* $dZ_t = -Z_t \sum_{i=1}^{d} b_t^i \, dW_t^i$.

Question 10.15. *Looks like Z is a martingale. Why did we assume it in Theorem 10.11?*

$$Z_t = \exp\Big(-X_t - \frac{1}{2}\int_0^t |b_s|^2 ds\Big) \qquad \Big(|b_s|^2 = \sum (b_s^i)^2\Big)$$

where $X_t = +\int_0^t b_s \cdot dW_s = \sum_{i=1}^{d} \int_0^t b^i \, dW^i$

$$f(t,x) = \exp\Big(-x - \frac{1}{2}\int_0^t |b_s|^2 ds\Big)$$

$$Z_t = f(t, X_t).$$

① $\partial_t f = \exp(\quad) \cdot \left(-\frac{1}{2} |b_t|^2\right)$

② $\partial_x f = \exp(\quad)(-1)$

③ $\partial_x^2 f = \exp(\quad)(+1)$

④ $d[X, X] = \sum_{i=1}^{d} \sum_{j=1}^{d} b_t^i b_t^j \, d[W_t^i, W_t^j]$ $\left(dX = \sum b_t^i \, dW_t^i\right.$

$$= \sum_{i=1}^{d} (b_t^i)^2 \, dt = |b_t|^2 \, dt$$

$$\Rightarrow dZ = \partial_t f \, dt + \partial_x f \, dX + \frac{1}{2} \partial^2_{x} f \, d[X,X]$$

$$= \exp(\;)\left[\left(-\frac{1}{2}|\sigma|^2 dt\right) - dX + \frac{1}{2}(1)(|\sigma_t|^2 dt)\right]$$

$$\boxed{\Rightarrow dZ_t = -Z_t \, \sigma_t \cdot dW_t} \longrightarrow M_g$$

Idea behind the proof of Theorem 10.11.

NTS $\tilde{W}$ is a BM under $\tilde{P}$.

Will Show $\boxed{① \ [\tilde{W}, \tilde{W}]_t = [W, W]_t = t}$ True

② $\tilde{W}$ is a $\tilde{P}$ mg — Needs checking (Use Lemma 10.10)

① & ② + Levy $\Rightarrow$ $\tilde{W}$ is a BM under $\tilde{P}$

Theorem (Theorem 10.4)**.** *The (unique) risk neutral measure is given by* $d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}$*, where*

$$Z_T = \exp\Big(-\int_0^T \theta_t \, dW_t - \frac{1}{2}\int_0^T \theta_t^2 \, dt\Big), \qquad \theta_t = \frac{\alpha_t - R_t}{\sigma_t}.$$

Proof of Theorem 10.4.

Know: $d\tilde{W} = (\dots)\,dt + dW \to$ Girsanov gives $\tilde{P}$ under which $\tilde{W}$ is a BM. (& has formula)

RNM. Want $D_t S_t$ to be a $\tilde{P}$ mg.

Compute $d(D_t S_t)$;

$$dS_t = \alpha_t S_t \, dt + \sigma_t S_t \, dW_t$$

$$dD_t = -R_t D_t \, dt$$

$$\Rightarrow d(D_t S_t) = D_t \, dS_t + S \, dD_t + \underbrace{d[S, D]}_{0}$$

$$= D_t(\alpha_t S_t \, dt + \sigma_t S_t \, dW) - R_t S_t D_t \, dt$$

$$= D_t \sigma_t S_t \left(\left(\frac{\alpha_t - R_t}{\sigma_t} \right) dt + dW_t \right)$$

$$= D_t \sigma_t S_t \left(\underline{\theta_t} \, dt + \underline{\underline{dW}} \right),$$

$$\boxed{\theta_t = \frac{\alpha_t - R_t}{\sigma_t}}$$

Market price of risk eq

$$= \underline{D_t \sigma_t S_t \, d\widetilde{W}_t},$$

By Girsanov $\widetilde{W}$ is a BM under $\widetilde{P}$

$$\boxed{d\widetilde{W} = \theta_t \, dt + dW}$$

where $d\tilde{P} = Z_T \, dP$,

& $$Z_T = \exp\left(- \int_0^T \theta_s \, dW_s - \frac{1}{2} \int_0^T \theta_s^2 \, dt \right)$$

Hence, under $\tilde{P}$,

$$d(D_t S_t) = \underbrace{\sigma_t D_t S_t \, \underbrace{d\tilde{W}}_{\text{BM under } \tilde{P}}}_{\text{Mg under } \tilde{P} \text{ !!}}$$

Theorem 10.16. *X_t represents the wealth of a self-financing portfolio if and only if $D_t X_t$ is a $\tilde{\boldsymbol{P}}$ martingale.*

Remark 10.17. The proof of the backward direction requires the *martingale representation theorem*, and is outlined on your homework.

Remark 10.18. This is the analog of Theorem 4.57 ← Same result for Binom Model.

Proof of the forward direction.

Assume X = wealth of a self fin port.

WTS: $D_t X_t$ is a $\tilde{P}$ mg.

Pf: By assumption $dX = \Delta_t \, dS_t + R_t (X_t - \Delta_t S_t) \, dt$

Self fin condition.

$$d(D_t X_t) = D_t\, dX_t + X_t\, dP_t + \underbrace{d[D, X]_t}_{0}$$

$$= -R_t D_t X_t + D_t\left(\Delta_t\, dS + R_t (X_t - \Delta_t S_t)\, dt\right)$$

$$= D_t \Delta_t\, dS - D_t R_t \Delta_t S\, dt \quad \cdots \quad (*)$$

Also note
$$d(D_t S_t) = D_t\, dS + S\, dD_t + 0$$
$$= D_t\, dS - R D S\, dt$$

Then (*) $\Rightarrow$
$$d(D_t X_t) = \Delta_t \left(D_t\, dS - R_t D_t S_t\, dt \right)$$
$$= \Delta_t\, \underbrace{d(D_t S_t)}_{\tilde{P}\ \text{mg!!}}$$

Here $D_t X_t$ is a $\hat{P}$ mg !!

Theorem (Theorem 10.5)**.** *Any security can be replicated. If a security pays V_T at time T, then the arbitrage free price at time t is*

$$V_t = \frac{1}{D_t}\tilde{\boldsymbol{E}}_t(D_T V_T) = \tilde{\boldsymbol{E}}_t\Big(\exp\Big(\int_t^T -R_s\,ds\Big)\,V_T\Big)\Big).$$

Remark 10.19. This is the analog of Proposition 4.1. ← Binom model

Proof of Theorem 10.5.

① Let $X_t = \frac{1}{D_t}\tilde{E}_t(D_T V_T)$

$\Rightarrow$ ⓐ $X_T = \frac{1}{D_T}\tilde{E}_T(D_T V_T) = V_T = \text{payoff}$

& (b) Compute $\tilde{E}_s(D_t X_t)$

$$= \tilde{E}_s\left(\tilde{E}_t(D_T X_T)\right)$$

$\underline{\text{tower}}$ $\tilde{E}_s(D_T X_T) = D_s X_s$

$\Rightarrow D_t X_t$ is a $\tilde{P}$ mg $\Rightarrow X =$ wealth of a self fin Port.

$\Rightarrow$ AFP at time $t = V_t = X_t = \dfrac{1}{D_t}\tilde{E}_t\left(D_T V_T\right)$

$$D_t = \exp\left(-\int_0^t R_s\,ds\right)$$

11. Black Scholes Formula revisited

- Suppose the interest rate $R_t = r$ (is constant in time).
- Suppose the price of the stock is a GBM(α, σ) (both α, σ are constant in time).

Theorem 11.1. *Consider a security that pays $V_T = g(S_T)$ at maturity time T. The arbitrage free price of this security at any time $t \leqslant T$ is given by $f(t, S_t)$, where*

$$f(t,x) = \int_{-\infty}^{\infty} e^{-r\tau} g\Big(x \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)\tau + \sigma\sqrt{\tau}\, y\Big)\Big) \frac{e^{-y^2/2}dy}{\sqrt{2\pi}}\,, \qquad \tau = T - t\,. \tag{8.4}$$

Remark 11.2. This proves Proposition 8.8.

Pf: ① $dS_t = \alpha S_t\, dt + \sigma S_t\, dW$ (GBM)

② Under RNM:

$$dS_t = r S_t \, dt + \sigma S_t \, d\tilde{W}$$

$$\Rightarrow S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right) t + \sigma \tilde{W}_t \right)$$

(From HW).

④ Compute $V_t \overset{RNP}{=} \frac{1}{D_t} \tilde{E}_t\left(D_T \, g(S_T)\right)$

$$= \frac{1}{e^{-rt}} \tilde{E}_t\left(e^{-rT} g(S_T)\right)$$

$$= e^{-r\tau} \tilde{E}_t\, g(S_T)$$

$$\left.\begin{aligned} S_T &= S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma \tilde{W}_T\right) \\ S_t &= S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma \tilde{W}_t\right) \end{aligned}\right\} \quad S_T = S_t \exp\left(\left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\left(\tilde{W}_T - \tilde{W}_t\right)\right)$$

$$\Longrightarrow V_t = e^{-r\tau}\,\tilde{E}_t\, g\Big(\underbrace{S_t}_{\mathcal{F}_t\text{ meas}} \exp\Big(\big(r-\tfrac{\sigma^2}{2}\big)\tau + \sigma\sqrt{\tau}\,\underbrace{\Big(\frac{\tilde{W}_T-\tilde{W}_t}{\sqrt{\tau}}\Big)}_{\text{indep of }\mathcal{F}_t}\Big)\Big)$$

$$\underset{\text{indep lemma}}{=} e^{-r\tau}\int_{y=-\infty}^{\infty} g\Big(\underbrace{S_t}_{x}\exp\Big(\big(r-\tfrac{\sigma^2}{2}\big)\tau+\sigma\sqrt{\tau}\,y\Big)\Big)\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy$$

QED.

Theorem 11.3 (Black Scholes Formula)**.** *The arbitrage free price of a European call with strike K and maturity T is given by:*

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)) \tag{8.5}$$

where

$$d_\pm(\tau,x) \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big), \tag{8.6}$$

and

$$N(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy, \tag{8.7}$$

is the CDF of a standard normal variable.

Just substitute. & simplify from Thm 11.2.

Remark 11.4. This proves Corollary 8.9.