

Stochastic Calculus for Finance I: Midterm.

2021-11-16, Pittsburgh

- *This is a closed book test. You may use a calculator. You may not give or receive assistance.*
- *Your calculator must not be able to access the internet, or store/read document files (PDF, word, etc.)*
- *You have 90 minutes. The exam has a total of 5 questions and 25 points.*
- *The questions are roughly ordered by difficulty. Good luck.*

In this exam W always denotes a standard Brownian motion, and the filtration $\{\mathcal{F}_t | t \geq 0\}$ (if not otherwise specified) is the Brownian filtration.

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 1. Find the Itô decomposition of the process $X_t = e^{-tW_t^2}$. That is, write $X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s$, and explicitly find X_0 , b and σ .
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 2. Find a (non-random) function $g = g(t, x)$ such that the process $M_t = W_t^3 + \int_0^t g(s, W_s) ds$ is a martingale.
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 3. Consider a discrete time market consisting of a bank and a stock. The bank pays interest rate $r = 5\%$ at every time period. Let S_n denote the stock price at time n , and we know $S_0 = \$10$. The stock price changes according to the flip of a fair coin: if the coin lands heads the stock price increases by 10% (i.e. $S_{n+1} = 1.1S_n$), and if the coin lands tails the stock price decreases by 5% (i.e. $S_{n+1} = 0.95S_n$). An option pays the holder S_N^3 at time $N = 5$. Find the arbitrage free price of this option at time $n = 1$. Also find the number of shares held in the replicating portfolio at time $n = 0$. Round your final answer two decimal places. (I recommend rounding intermediate steps to three decimal places.)
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 4. Compute $\mathbf{E} \left[\left(\int_0^t e^{-2s} dW_s \right)^4 \right]$. Express your final answer in terms of t without involving expectations or integrals.
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 5. Let $M_t = \int_0^t s W_s ds$. Find $\mathbf{E}(M_t^2 - [M, M]_t)$. Express your final answer in terms of t without involving expectations or integrals.