12. Review problems

Problem 12.1. Consider the N period Binomial model with $N = 5$, and parameters $0 < d < 1 + r < u$. At maturity $N = 5$, a security pays \$1 if $S_5 > (1 + r)S_4$, and 0 otherwise. Find the arbitrage free price and trading strategy trading at time 0.

What not to do?

LAST RESORT

Draw a tree of all coin tosses!

Better strategy: If payoff is a fn of Stock price then so is the AFP i.e. Drawing a tree of <u>Stock prices</u> may be enough

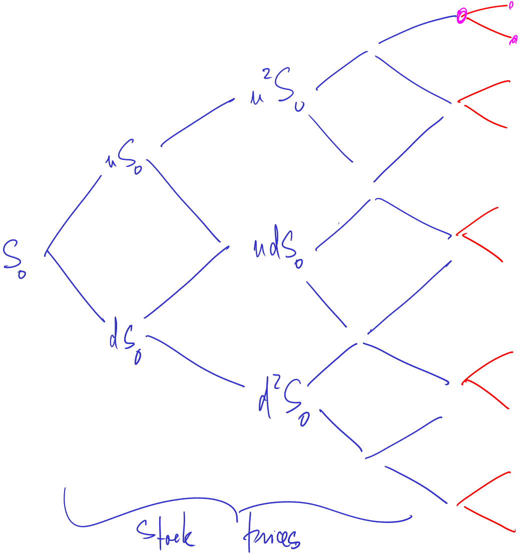
(Much much less work!!)

Draw tree of Stock prices up to time 4
& then coin tosses

$$\tilde{p} = \frac{1+r-d}{u-d}$$

$$\tilde{q} = \frac{u-(1+r)}{u-d}$$

$$V_4 = \frac{1}{1+r}\tilde{E}_4 V_5$$



$V_5 = 1$

0

1

0

1

0

1

0

1

0

$V_4 = \tilde{p}/1+r$

$= \tilde{p}/1+r.$

$\tilde{p}/1+r$

⋮

time $n=5$

draws coin tosses

$$V_0 = \frac{1}{(1+r)^4}\tilde{E}\, V_4 = \frac{\tilde{p}/1+r}{(1+r)^4} = \frac{\tilde{p}}{(1+r)^5}$$

Trading strat

$$\Delta_0 = \frac{V_1(1) - V_1(-1)}{S_0(u-d)} = 0$$

Problem 12.2. Let $\boxed{f}$ be a deterministic function, and define

$$X_t \overset{\text{def}}{=} \underbrace{\int_0^t f(s) W_s \, ds}.$$

Find the distribution of X.

think of this as

$\lim_{\|P\| \to 0} \underbrace{\sum f(t_i) W_{t_i} (t_{i+1} - t_i)}$

l.c. of normals

$\longrightarrow$ Normal.

$X \sim N \Big($

$EX_t = E \int_0^{\boxed{t}} f(s) W_s \, ds = 0$

② $$E X_t^2 = E\left(\int_0^t f(s)\, W_s\, ds\right)^2$$

$$= E \int_0^t f(s)\, W_s\, ds \int_0^t f(\underset{r}{\underset{\downarrow}{s}})\, W_{\underset{r}{\underset{\downarrow}{s}}}\, d\underset{r}{\underset{\downarrow}{s}}$$

$$= E \int_0^t \int_0^t f(s) f(r)\, W_s W_r\, ds\, dr$$

$$= \int_0^t \int_0^t f(s) f(r) \, (s \wedge r) \, ds \, dr$$

$$= \int_{r=0}^t \int_{s=0}^r f(s) f(r) \, s \, ds \, dr + \int_{r=0}^t \int_{s=r}^t f(s) f(r) \, r \, ds \, dr$$

& evaluate both!

Problem 12.3. Suppose σ, τ, ρ are three deterministic functions and M and N are two continuous martingales with respect to a common filtration $\{\mathcal{F}_t\}$ such that $M_0 = N_0 = 0$, and

$$d[M,M]_t = \overset{\sigma(t)}{\cancel{\sigma_t}}\,dt\,, \qquad d[N,N]_t = \overset{\tau(t)}{\cancel{\tau_t}}\,dt\,, \qquad \text{and} \qquad d[M,N]_t = \overset{\rho(t)}{\cancel{\rho_t}}\,dt\,.$$

(a) Compute the joint moment generating function $\boxed{\boldsymbol{E}\exp(\lambda M(t) + \mu N(t))}$.

(b) *(Lévy's criterion)* If $\sigma = \tau = 1$ and $\rho = 0$, show that (M, N) is a two dimensional Brownian motion.

Let $\varphi(t) = E\, e^{\lambda M_t + \mu N_t}$.

Find $\varphi(t)$: Let $X_t = e^{\lambda M_t + \mu N_t} = \overbrace{f(t, M_t, N_t)}$

where $f(t,x,y) = e^{\lambda x + \mu y}$

Compute

1. $\partial_t f = 0$
2. $\partial_x f = \lambda f \quad , \quad \partial_x^2 f = \lambda^2 f$
3. $\partial_y f = \mu f \quad , \quad \partial_y^2 f = \mu^2 f$
4. $\partial_x \partial_y f = \lambda \mu f.$

$$\Rightarrow dX_t = \partial_t f \, dt + \partial_x f \, dM + \partial_y f \, dN + $$

$$+ \frac{1}{2}\Big(\partial_x^2 f \, d[M,M] + \partial_y^2 f \, d[N,N]$$

$$\partial_x \partial_y f \, d[M,N] + \partial_y \partial_x f \, d[N,M]$$

equal.

$$= 0 + \lambda f \, dM + \mu f \, dN$$

$$+ \frac{1}{2}\Big(\lambda^2 f \, \tau + \mu^2 f \, \tau + 2 \lambda \mu \, \rho f \Big) dt$$

$$\Rightarrow \quad e^{\lambda M_t + \mu N_t} - 1 = X_t - X_0$$

$$= \underbrace{\lambda \int_0^t \overset{X}{\cancel{f}}\, dM_s + \mu \int_0^t \overset{X}{\cancel{f}}\, dN_s + \frac{1}{2} \int_0^t \overset{X}{\cancel{f}} \overbrace{\left(\lambda^2 \sigma + \mu^2 \tau + 2\lambda\mu\rho\right)}^{\text{Not random.}} ds}$$

$$\Rightarrow \quad E\left(e^{\lambda M_t - \mu N_t} - 1\right) = EX_t - X_0 = \varphi(t) - 1$$

$$= 0 + 0 + \frac{1}{2} \int_0^t \varphi(s) \left(\lambda^2 \sigma(s) + \mu^2 \tau(s) + 2\lambda\mu\, \rho(s)\right) ds.$$

$$\Rightarrow \partial_t \varphi = \frac{1}{2}\left(\lambda^2 \sigma(t) + \mu^2 \tau(t) + 2\lambda\mu\,\rho(t)\right)\varphi(t)$$

Solve $\Rightarrow$ $$\varphi(t) = 1 \cdot \exp\left(\frac{1}{2}\left[\lambda^2 \int_0^t \sigma\,ds + \mu^2 \int_0^t \tau + 2\lambda\mu \int_0^t \rho\right]\right)$$

= MGF of normal with mean 0

& cov matrix $\begin{pmatrix} \int_0^t \sigma & \int_0^t \rho \\ \int_0^t \rho & \int_0^t \tau \end{pmatrix}$ ... $(*)$

Note: If $\sigma = 1$, $\tau = 1$, $\rho = 0$

then
$$\left.\begin{aligned} d[M,M] &= dt \\ d[N,N] &= dt \\ d[M,N] &= 0 \end{aligned}\right\} \leftarrow \text{Joint QV of a 2D Br. M.}$$

(*) $\Rightarrow$ MGF of (M, N) = MGF of Normal, mean 0
& Cov matrix $\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$

Can check similarly

$$\begin{pmatrix} M_t - M_s \\ N_t - N_s \end{pmatrix} \sim \text{2D normal mean } 0 \text{ \& cov } \begin{pmatrix} t-s & 0 \\ 0 & t-s \end{pmatrix}$$

& is ind of $\mathcal{F}_s$

$\Rightarrow (M, N)$ is a 2D B.M. (Levy!)

Problem 12.4. Consider a financial market consisting of a $\boxed{\text{risky asset}}$ and a money market account. Suppose the return rate on the money market account is r, and the price of the risky asset, denoted by S, is a geometric Brownian motion with mean return rate α and volatility σ. Here r, α and σ are all deterministic constants. Compute the arbitrage free price of derivative security that pays

$$\longrightarrow \quad V_T = \frac{1}{T}\int_0^T S_t \, dt$$

at maturity T. Also compute the trading strategy in the replicating portfolio.

RNP formula: For any $t \le T$,

$$D_t = \exp\left(-\int_0^t R_s \, ds\right) = e^{-rt}$$

AFP is $V_t = \frac{1}{D_t}\tilde{E}_t(D_T V_T)$

$$= e^{-r(T-t)}\, \tilde{E}_t V_T$$

$$= e^{-r(T-t)}\, \tilde{E}_t \frac{1}{T}\int_0^T S_s\, ds \qquad (\tau = T-t)$$

$$= \frac{e^{-r\tau}}{T}\left(\int_0^t \tilde{E}_t S_s\, ds + \int_t^T \tilde{E}_t S_s\, ds\right)$$

$$= \frac{e^{-rT}}{T}\left(\int_0^t S_s\,ds + \int_t^T \underbrace{\tilde{E}_t S_s}\,ds\right)$$

Need to find this!

(s>t)

Option 1: $S_s = S_t \exp\left(\left(r - \frac{\sigma^2}{2}\right)(s-t) + \sigma\left(\tilde{W}_s - \tilde{W}_t\right)\right)$

($dS = \alpha S\,dt + \sigma S\,dW$

apply Ito to $\ln(S)$ & solve)

$$\hookrightarrow \Rightarrow \tilde{E}_t S_s = \tilde{E}_t\left(S_t \exp\left(\left(r - \frac{\sigma^2}{2}\right)(s-t) + \sigma\,\underbrace{(\tilde{W}_s - \tilde{W}_t)}_{\hookrightarrow N(0,\,s-t)}\right)\right)$$

(option 1)

indep lemma

$$\int_{y=-\infty}^{\infty}\left(S_t \exp\left(\left(r - \frac{\sigma^2}{2}\right)(s-t) + \sigma\sqrt{s-t}\;y\right)\right)\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,dy$$

(option 2)

$$= S_t\, e^{\left(r - \sigma^2/2\right)(s-t)}\;\tilde{E}\, e^{\sigma\left(\tilde{W}_s - \tilde{W}_t\right)}$$

$$= S_t \, e^{(r - \sigma^2/2)(s-t)} \, e^{\sigma^2 (s-t)/2}$$

(MGF of normal!)

$(s > t)$

Option 2: Find $\tilde{E}_t \, \tilde{S}_s$ by discounting! (Much faster)

Know $e^{-rs} S_s$ is a $\tilde{P}$ mg!

$$\Rightarrow \tilde{E}_t S_s = e^{+rs}\tilde{E}_t\left(e^{-rs}S_s\right) \overset{\text{mg}}{=} e^{+rs}\left(e^{-rt}S_t\right)$$

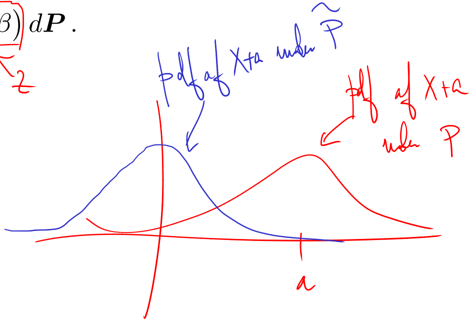
$$= e^{r(s-t)}S_t\ !$$

Substitute back & get V_t!

Problem 12.5. Let $\boxed{X \sim N(0,1)}$, and $a, \alpha, \beta \in \mathbb{R}$. Define a new measure $\tilde{\boldsymbol{P}}$ by

$$d\tilde{\boldsymbol{P}} = \underbrace{\exp(\alpha X + \beta)}_{Z}\, d\boldsymbol{P}\,.$$

Find α, β such that $\boxed{X + a} \sim N(0,1)$ under $\tilde{\boldsymbol{P}}$.



Compute MGF of $X+a$ under $\tilde{P}$

Find $\tilde{E}\left(e^{\lambda(X+a)}\right)$

Note $\tilde{E}\left(e^{\lambda(X+a)}\right) = E\left(e^{\lambda(X+a)} \cdot Z\right)$

$$= \mathbb{E}\left(e^{\lambda(X+a) + \alpha X + \beta} \right)$$

$$= \left(\mathbb{E}\, e^{(\lambda+\alpha)X} \right) \cdot e^{a\lambda + \beta}$$

$$= e^{(\lambda+\alpha)^2/2 + a\lambda + \beta}$$

Want

$$= e^{\lambda^2/2}.$$

Set $\dfrac{(\lambda+\alpha)^2}{2} + a\lambda + \beta = \dfrac{\lambda^2}{2}$ (for all $\lambda \in \mathbb{R}$)

$$\Leftrightarrow \frac{\lambda^2}{2} + \alpha\lambda + \frac{\alpha^2}{2} + a\lambda + \beta = \frac{\lambda^2}{2} \quad (\qquad)$$

$$\Leftrightarrow \underbrace{(\alpha+a)}_{\text{equate to } 0}\lambda + \underbrace{\frac{\alpha^2}{2} + \beta}_{\text{equate to } 0} = 0$$

$$\boxed{\begin{aligned} \text{set } \alpha &= -a \\ \beta &= -\frac{a^2}{2} \end{aligned}}$$

Problem 12.6. Let $x_0, \mu, \theta, \sigma \in \mathbb{R}$, and suppose X is an Itô process that satisfies

$$dX(t) = \theta(\mu - X_t)\, dt + \sigma\, dW_t\,,$$

with $X_0 = x_0$.

(a) Find functions $f = f(t)$ and $g = g(s,t)$ such that

$$X(t) = f(t) + \int_0^t g(s,t)\, dW_s\,.$$

The functions f, g may depend on the parameters x_0, θ, μ and σ, but should not depend on X.

(b) Compute $\boldsymbol{E} X_t$ and $\operatorname{cov}(X_s, X_t)$ explicitly.

Problem 12.7. Let $\theta \in \mathbb{R}$ and define

$$Z(t) = \exp\Big(\theta W_t - \frac{\theta^2 t}{2}\Big).$$

Given $0 \leqslant s < t$, and a function f, find a function such that

$$\boldsymbol{E}_s f(Z_t) = g(Z(s)).$$

Your formula for the function g can involve f, s, t and integrals, but not the process Z or expectations.

Problem 12.8. Let W be a Brownian motion, and define

$$B_t = \int_0^t \operatorname{sign}(W_s)\, dW_s \,.$$

$$\left(\operatorname{sign}(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases} \right)$$

(a) Show that B is a Brownian motion.

(b) Is there an adapted process σ such that

$$W_t = \int_0^t \sigma_s \, dB_s \,?$$

If yes, find it. If no, explain why.

(c) Compute the joint quadratic variation $[B, W]$.

(d) Are B and W uncorrelated? Are they independent? Justify.

(a) $\longrightarrow$ Levy: Need

① Continuous

② B is a mg (Ito int ✓)

③ $d[B,B] = dt$ ($\because d[B,B] = 1\, dt$ ✓)

(b) Find σ s.t. that $W_t = \int_0^t \sigma_s \, dB_s$.

Note $dB = \overbrace{\text{sign}(W_s)} \, dW_s$

$\Rightarrow \quad dW_s = \underbrace{\frac{1}{\text{sign}(W_s)}}_{\sigma_s} \, dB_s$

$$\boxed{\Rightarrow W_t = \int_0^t \text{sign}(W_s) \, dB_s}$$

$$dB_t = \overbrace{\text{sign}(W_t)}\, dW_t$$

$$dW_t = \underbrace{1}\, dW_t$$

ⓒ $$d[B, W]_t = \text{sign}(W_t) \cdot 1 \; d[W, W]_t$$

$$= \text{sign}(W_t)\, dt$$

$$\Rightarrow \quad [B, W]_t = \int_0^t \text{sign}(W_s)\, \cancel{dW_s}\, dt$$

ⓓ Find $E(B_t W_t)$:

$$d(BW) = B\,dW + W\,dB + d[B,W]$$

$$\Rightarrow B_t W_t - 0 = \int_0^t B_s\,dW_s + \int_0^t W_s\,dB_s + \int_0^t \operatorname{sign}(W_s)\,ds$$

$$\Rightarrow E(B_t W_t) = 0 + 0 + \int_0^t \underbrace{E\,\operatorname{sign}(W_s)}_{0}\,ds$$

$\Rightarrow$ B & W are uncorrelated!

Note B_t & W_t are both normal, Not Jointly normal! need not $\Rightarrow$ indep

If B & W were indep $\Rightarrow [B, W] = 0$

But $[B, W] = \int_0^t \operatorname{Sign}(W_s)\, ds \neq 0$

$\Rightarrow$ B & W can NOT be indep!

Problem 12.9. Let W be a Brownian motion. Does there exist an equivalent measure $\tilde{\boldsymbol{P}}$ under which the process tW_t is a Brownian motion? Prove it.

$$d\left(\int_0^t f(s)\,ds\right) = f(t)\,dt$$

$$\frac{d}{dt}\int_0^t f(s)\,ds = f(t)$$

$$M_t = \int_0^t W_s\,ds$$

Find EM_t^2.

$$d(M^2) = \underbrace{2\,M\,dM}_{\text{not mg!}} + \underbrace{d\,[M, M]}_{0}$$

$$M_t^2 = 2\int_0^t \underbrace{M_s W_s}\, ds + 0$$

$$d(MW)_t = M_t\, dW_t + W_t^2\, dt \Rightarrow E(M_t W_t) = 0 + \int_0^t s\, ds$$