Just unmute if you have questions. (I can't see the chat)

① $W_t - W_s$ ind of $\mathcal{F}_s$

② $W_t - W_s \sim N(0, t-s)$

$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ $P(W_0 = 0)$ $\Rightarrow W_t \sim N(0, t)$

Q: Find ?

$\tilde{E}(S_t S_s) \longrightarrow$

$S = GBM(\alpha, \sigma)$ under P

$S = GBM(r, \sigma)$ under $\tilde{P}$.

$$\tilde{E}(S_t S_s) = \tilde{E}\,\tilde{E}_s(S_s S_t)$$

$$= \tilde{E}\left(\frac{S_s}{D_t}\,\tilde{E}_s(D_t S_t)\right)$$

$D_t = e^{-rt}$

$$= \tilde{E}\left(\frac{S_s}{D_t}\,D_s S_s\right) = \tilde{E}\left(S_s^2 \cdot e^{r(t-s)}\right)$$

$$= \$\, e^{-r(t-s)}\, \tilde{E}\, S_0^2 \exp\left(2 \left(r - \frac{\sigma^2}{2} \right) s + 2\sigma \tilde{W}_s \right)$$

& use mgf of normal.

B.S. $\rightarrow$ call strike K, mat T, int rate r.

$$\text{AFP} = \boxed{c(t,x) = x\,N(d_+) - Ke^{-r\tau}N(d_-)} \qquad \tau = T-t$$

$$\text{RNP}: \quad c(t, S_t) = e^{-r\tau}\tilde{E}_t\,(S_T - K)^+$$

(S_t is a GBM(r, σ) under $\tilde{P}$).

$$Q2: \text{AFP} = e^{-r\tau}\underbrace{\tilde{E}_t\,(S_T^2 - K)^+}$$

Compute $d(S_t^2) = 2S_t\,dS + d[S, S]$ (under $\tilde{P}$)

$$= 2S_t\left(rS_t\,dt + \sigma S_t\,d\widetilde{W}\right) + \sigma^2 S^2\,dt$$

$$d(S_t^2) = (2r+\sigma^2)\,S_t^2\,dt + 2\sigma\,S_t^2\,d\widetilde{W}$$

$\Rightarrow$ S_t^2 is a GBM$(2r+\sigma^2,\ 2\sigma)$

Q1) $M_t = \int_0^t W_s \, dB_s$ $\qquad$ $N_t = \int_0^t B_s \, dW_s$

$$d[M,M]_t = W_t^2 \, d[B,B]_t = W_t^2 \, dt$$

$$d[N,N]_t = B_t^2 \, d[W,W]_t = B_t^2 \, dt$$

$$d[M,N]_t = W_t B_t \, d[W,B] = 0 \qquad (\because W \,\&\, B \text{ are ind})$$

2019 Q3: $E\left(W_s \, |W_t|\right)$

$$E\, W_s |W_s| = \int_{-\infty}^{\infty} x|x| \, e^{-x^2/2s} \frac{dx}{\sqrt{2\pi s}} = 0$$

2019 Q5

$$X_t = (W_t + t^2) M_t$$

find M so that X is a mg

$\rightarrow \underline{\underline{\lim_{t \to 0}}} \left(\frac{1}{t} \int_0^t X_s \, ds \right) = X_0$ (FTC)

$$\frac{t}{t}\int_0^t X_s \, ds$$

Answers to 2016 Final:

1. $$W_s^3 \int_0^s (r+W_r)^2 \, dW_r$$

2. $$X_t = \int_0^t \left(3e^{3s}W_s^2 + e^{3s}\right) ds + \int_0^t 2e^{3s} W_s \, dW_s$$

3. $$\sigma_s = \left(1 - \frac{1}{\left(1+(W_s')^2\right)^2}\right)^{1/2}$$

4. $$EX_t = \int_0^t e^{9s/2} \, ds$$

⑤ True ($\Gamma_t = p(t, S_t) - S\,\partial_x p(t, S_t)$

Simplifies to $K e^{-r(T-t)} (N(d_-)) > 0$)

⑥ $\left(\frac{t^3}{3}\right)^{1/4} \int |x|^{1/2} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}$ (Use $\int_0^t W_s\,ds$ is normal)

⑦ $$V_t = \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{\mathbb{R}} \left[\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\sqrt{\tau}\,y + \ln\left(\frac{S_t}{S_0}\right)\right]^+ e^{-y^2/2}\,dy$$