- $X_t = (X_t^1, \cdots, X_t^d)$ is an Ito's process

$f(t,x) = f(t, x^1, \cdots, x^d)$ is $C^{1,2}$ (C^1 in t , C^2 in x)

$$df(t, X_t) = \partial_t f(t, X_t)\, dt + \sum_{i=1}^{d} \partial_i f(t, X_t)\, dX_t^i + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \partial_i \partial_j f(t, X_t)\, d[X^i, X^j]_t$$

If $dX_t^i = b_t^i\, dt + \sigma_t^i\, dW_t$, then $d[X^i, X^j]_t = \sigma_t^i \sigma_t^j\, dt$.

Ex. 9.23

$f \in C^{1,2}$, W d-dimensional BM , $X_t = f(t, W_t)$

$$dX_t = \partial_t f(t, W_t)\, dt + \sum_{j=1}^{d} \partial_i f(t, W_t)\, dW_t^i + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \partial_i \partial_j f(t, W_t)\, d[W^i, W^j]_t$$

$$d[W^i, W^j]_t = \begin{cases} dt , & i = j \\ 0 & i \neq j \end{cases}$$

$$dX_t = \partial_t f(t, W_t)\, dt + \sum_{i=1}^{d} \partial_i f(t, W_t)\, dW_t^i + \frac{1}{2} \sum_{i=1}^{d} \partial_i^2 f(t, W_t)\, dt .$$

$$= (\; --- \;)\, dt + \sum_{i=1}^{d} (\; --- \;)\, dW_t^i .$$

Ex. $M_t = \left(W_t + \frac{t^2}{2}\right) \exp\left(-\int_0^t s\, dW_s - \frac{t^3}{6}\right)$. Is M a martingale?

Method 1:

$f(t,x) = \left(x + \frac{t^2}{2}\right) \exp\left(-\int_0^t s\, dW_s - \frac{t^3}{6}\right)$. WRONG.

f is not $C^{1,2}$.

$d(t W_t) = t\, dW_t + W_t\, dt \Rightarrow t W_t = \int_0^t s\, dW_s + \int_0^t W_s\, ds$

$\Rightarrow \int_0^t s\, dW_s = t W_t - \int_0^t W_s\, ds$.

$M_t = \left(W_t + \frac{t^2}{2}\right) \exp\left(\int_0^t W_s\, ds - t W_t - \frac{t^3}{6}\right)$

$f(t,x) = \left(x + \frac{t^2}{2}\right) \exp\left(\int_0^t W_s\, ds - t x - \frac{t^3}{6}\right)$.

$\partial_t f$, $\partial_x f$, $\partial_x^2 f$ ...

Method 2:

$X_t^1 = W_t$, $X_t^2 = \int_0^t s\, dW_s$, $\quad (W_t = \int_0^t 1\, dW_s)$

$d[X^1, X^1]_t = dt$, $d[X^2, X^2]_t = t^2 dt$, $d[X^1, X^2]_t = t\, dt$

$\cancel{f(t,x) = (x^1 + \frac{t^2}{2}) \exp(-x^2 - \frac{t^3}{6})}$

$f(t, x, y) = (x + \frac{t^2}{2}) \exp(-y - \frac{t^3}{6})$.

$\partial_t f = t \exp(-y - \frac{t^3}{6}) + (x + \frac{t^2}{2}) \exp(-y - \frac{t^3}{6}) (-\frac{t^2}{2})$

$\partial_x f = 1 \cdot \exp(-y - \frac{t^3}{6})$

$\partial_y f = -(x + \frac{t^2}{2}) \exp(-y - \frac{t^3}{6})$

$\partial_x^2 f = 0$, $\partial_y^2 f = (x + \frac{t^2}{2}) \exp(-y - \frac{t^3}{6})$, $\partial_x \partial_y f = -\exp(-y - \frac{t^3}{6})$.

$dM_t = df(t, X_t^1, X_t^2) = \partial_t f\, dt + \partial_1 f\, dX_t^1 + \partial_2 f\, dX_t^2 + \frac{1}{2} \partial_1^2 f\, d[X^1, X^1]_t$

$+ \frac{1}{2} \partial_2^2 f\, d[X^2, X^2]_t + \partial_1 \partial_2 f\, d[X^1, X^2]_t$

$(\frac{1}{2} \partial_1 \partial_2 f + \frac{1}{2} \partial_2 \partial_1 f)$

$$dM_t = \exp(\cdots)\left[t - \frac{t^2}{2}\left(X_t^1 + \frac{t^2}{2}\right)\right] dt + \exp(\cdots)\, dW_t + \exp(\cdots)\left(-\left(X_t^1 + \frac{t^2}{2}\right)\right) t\, dW_t$$

$$+ \frac{1}{2}\exp(\cdots)\left(X_t^1 + \frac{t^2}{2}\right) \overset{t^2}{} dt + (-1)\exp(\cdots)\, t\, dt$$

$$= (\cdots)\, dW_t + \exp(\cdots)\left[\cancel{t} - \frac{t^2}{2}\cancel{\left(W_t + \frac{t^2}{2}\right)} + \frac{1}{2}\cancel{\left(W_t + \frac{t^2}{2}\right) t^2} - \cancel{t}\right] dt$$

$$= (\cdots)\, dW_t$$

Ex. 8.24 W 2-dim BM, $X_t = \ln\left((W_t^1)^2 + (W_t^2)^2\right)$, $t > 0$

Is X a martingale?

$$f(t, x, y) = \ln(x^2 + y^2)$$

$$\partial_t f = 0, \quad \partial_x f = \frac{2x}{x^2+y^2}, \quad \partial_y f = \frac{2y}{x^2+y^2}$$

$$\partial_x^2 f = \frac{2(x^2+y^2) - 2x \cdot 2x}{(x^2+y^2)^2} = \frac{2y^2 - 2x^2}{(x^2+y^2)^2}, \quad \partial_y^2 f = \frac{2x^2 - 2y^2}{(x^2+y^2)^2}$$

$$dX_t = \partial_t f\, dt + \partial_1 f\, dW_t^1 + \partial_2 f\, dW_t^2 + \frac{1}{2}\partial_1^2 f\, dt + \frac{1}{2}\partial_2^2 f\, dt + \partial_1\partial_2 f \cdot \times 0.$$

$$= (\cdots)\, dW_t^1 + (\cdots)\, dW_t^2 + \frac{1}{2}\underbrace{\left[\frac{2(W_t^2)^2 - 2(W_t^1)^2}{(\quad)^2} + \frac{2(W_t^1)^2 - 2(W_t^2)^2}{(\quad)^2}\right]}_{=0} dt$$

$$dX_t = (\cdots)\,dW_t^1 + (\cdots)\,dW_t^2 .$$

($M_t = \int_0^t \sigma_s \, dW_s$, if $\mathbb{E}\int_0^t \sigma_s^2 \, ds < \infty$, $\forall t$,

then M_t is a martingale.)

$$\mathbb{E} X_t = \mathbb{E} \ln\left((W_t^1)^2 + (W_t^2)^2 \right) \quad , (Z^1, Z^2) \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$= \mathbb{E} \ln\left((\sqrt{t}\, Z^1)^2 + (\sqrt{t}\, Z^2)^2 \right)$$

$$= \mathbb{E} \ln\left(t (Z^1)^2 + t (Z^2)^2 \right)$$

$$= \ln t + \mathbb{E} \ln\left((Z^1)^2 + (Z^2)^2 \right)$$ is not constant.

$$dX_t = \partial_1 f \, dW_t^1 + \partial_2 f \, dW_t^2$$

$$\mathbb{E}\left(\int_0^t |\partial_1 f|^2 ds + \int_0^t |\partial_2 f|^2 ds \right) = \infty .$$

- Girsanov theorem.

$$Z_t = \exp\left(-\int_0^t b_s \cdot dW_s - \frac{1}{2}\int_0^t |b_s|^2 ds\right) \text{ is a martingale,}$$

$$d\tilde{\mathbb{P}} = Z_T \, d\mathbb{P} \qquad (\Omega, \mathcal{F}_T)$$

For $A \in \mathcal{F}_T$, $\tilde{\mathbb{P}}(A) = \mathbb{E}(Z_T \mathbb{1}_A)$, $\int_A d\tilde{\mathbb{P}} = \int_A Z_T \, d\mathbb{P}$

X is r.v. $\tilde{\mathbb{E}} X = \mathbb{E}(Z_T X)$, $\int_\Omega X(\omega) d\tilde{\mathbb{P}}(\omega) = \int_\Omega Z_T(\omega) X(\omega) \, d\mathbb{P}(\omega)$.

Then, $\tilde{W}_t = W_t + \int_0^t b_s \, ds$, $t \in [0,T]$ is a BM under $\tilde{\mathbb{P}}$.

Ex. Compute $\mathbb{E}\left(e^{W_T} \int_0^T W_s^2 \, ds\right)$.

$b_t = -1$

$$Z_t = \exp\left(\int_0^t 1 \, dW_s - \frac{1}{2}\int_0^t 1 \, ds\right)$$

$$= \exp\left(W_t - \frac{1}{2}t\right) \text{ is a martingale.}$$

$$Z_T = \exp\left(W_T - \frac{1}{2}T\right).$$

$d\tilde{\mathbb{P}} = Z_T \, d\mathbb{P}$. $\tilde{W}_t = W_t - t$, $t \in [0,T]$ is a BM under $\tilde{\mathbb{P}}$.

$$\mathbb{E}\left(e^{W_T}\int_0^T W_s^2\,ds\right) = \mathbb{E}\left(\underline{e^{W_T-\frac{T}{2}}}\int_0^T W_s^2\,ds \cdot e^{\frac{T}{2}}\right)$$

$$= e^{\frac{T}{2}}\,\mathbb{E}\left(Z_T\int_0^T W_s^2\,ds\right) = e^{\frac{T}{2}}\,\widetilde{\mathbb{E}}\int_0^T W_s^2\,ds = e^{\frac{T}{2}}\,\widetilde{\mathbb{E}}\int_0^T(\widetilde{W}_s+s)^2\,ds$$

$$= e^{\frac{T}{2}}\int_0^T \widetilde{\mathbb{E}}\left(\widetilde{W}_s^2 + 2s\widetilde{W}_s + s^2\right)ds = e^{\frac{T}{2}}\int_0^T\left(s+0+s^2\right)ds = e^{\frac{T}{2}}\left(\frac{T^2}{2}+\frac{T^3}{3}\right).$$

Ex. Under $\widetilde{\mathbb{P}}$, ~~$dS_t =$~~ α, σ, r are constant.

$$dS_t = rS_t\,dt + \sigma S_t\,d\widetilde{W}_t.$$

pf. $dS_t = \alpha S_t\,dt + \sigma S_t\,dW_t \qquad (\widetilde{W}_t = W_t + \theta t)$

$$= \alpha S_t\,dt + \sigma S_t\,d(\widetilde{W}_t - \theta t)$$

$$= \alpha S_t\,dt + \sigma S_t\,d\widetilde{W}_t - \theta\sigma S_t\,dt$$

$$= (\alpha - \theta\sigma)S_t\,dt + \sigma S_t\,d\widetilde{W}_t$$

$$= \left(\alpha - \frac{\alpha - r}{\sigma}\cdot\sigma\right)S_t\,dt + \sigma S_t\,d\widetilde{W}_t$$

$$= rS_t\,dt + \sigma S_t\,d\widetilde{W}_t.$$

- The (unique) risk neutral measure is given by

$$d\tilde{\mathbb{P}} = \exp\left(-\int_0^T \theta_s \, dW_s - \frac{1}{2}\int_0^T \theta_s^2 \, ds\right) d\mathbb{P}, \quad \theta_t = \frac{\alpha_t - R_t}{\sigma_t}$$

$$dS_t = \alpha_t S_t \, dt + \sigma_t S_t \, dW_t, \quad D_t = e^{-\int_t^T R_s \, ds}.$$

($\tilde{W}_t = W_t + \int_0^t \theta_s \, ds$, $t \in [0, T]$ is a BM under $\tilde{\mathbb{P}}$).

- Theorem 10.5

A security pays V_T at maturity T.

AFP at time t is $V_t = \frac{1}{D_t} \tilde{\mathbb{E}}_t (D_T V_T)$

($D_t V_t$ is a martingale under $\tilde{\mathbb{P}}$).

- Theorem 11.1

A security pays $V_T = g(S_T)$ at T, AFP at t $f(t, S_t)$,

$$f(t, x) = \int_{-\infty}^{+\infty} e^{-r\tau} g\left(x \exp\left((r - \frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau} y\right)\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy, \quad \tau = T - t.$$

$$dS_t = rS_t\,dt + \sigma S_t\,d\widetilde{W}_t \Rightarrow S_t = S_0 \exp\left((r-\frac{\sigma^2}{2})t + \sigma\widetilde{W}_t\right).$$

$$S_T = S_0 \exp\left((r-\frac{\sigma^2}{2})T + \sigma\widetilde{W}_T\right)$$

$$\Rightarrow S_T = S_t \exp\left((r-\frac{\sigma^2}{2})\tau + \sigma(\widetilde{W}_T - \widetilde{W}_t)\right).$$

$$V_t = e^{rt}\,\widetilde{\mathbb{E}}_t\left(e^{-rT}V_T\right) = e^{-r\tau}\,\widetilde{\mathbb{E}}_t\left(g(S_T)\right)$$

$$= e^{-r\tau}\,\widetilde{\mathbb{E}}_t\left(g\left(\underline{S_t}\exp\left((r-\frac{\sigma^2}{2})\tau + \sigma(\underline{\widetilde{W}_T - \widetilde{W}_t})\right)\right)\right)$$

$$= f(t, S_t)$$

$$f(t,x) = e^{-r\tau}\,\widetilde{\mathbb{E}}\left(g\left(x\exp\left((r-\frac{\sigma^2}{2})\tau + \sigma(\widetilde{W}_T - \widetilde{W}_t)\right)\right)\right), \quad \widetilde{W}_T - \widetilde{W}_t \sim N(0,\tau).$$

$$= e^{-r\tau}\int_{-\infty}^{+\infty} g\left(x\exp\left((r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}\,y\right)\right)\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dy.$$

Ex. European call with strike K. $g(x) = (x-K)^+ = \max(x-K, 0)$.

$$c(t,x) = e^{-r\tau} \int_{-\infty}^{+\infty} \left(x \exp\left((r - \tfrac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}\, y \right) \cancel{\cdot \tfrac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}} - K \right)^+ \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy.$$

$$x^+ = \begin{cases} x, & x > 0 \\ 0, & x \le 0 \end{cases}$$

$$x \exp\left((r - \tfrac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}\, y \right) - K > 0$$

$$\exp\left((r - \tfrac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}\, y \right) > \frac{K}{x}$$

$$(r - \tfrac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}\, y > \ln\left(\frac{K}{x}\right) = -\ln\left(\frac{x}{K}\right)$$

$$y > \frac{-\ln\left(\frac{x}{K}\right) - (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} = -\frac{\ln\left(\frac{x}{K}\right) + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} = -d_-(\tau, x).$$

$$\Rightarrow c(t,x) = e^{-r\tau} \int_{-d_-}^{+\infty} \left(x \exp\left((r - \tfrac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}\, y \right) - K \right) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy$$

$$= \cancel{e^{-r\tau}}\, x \int_{-d_-}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left((\cancel{r} - \tfrac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}\, y - \tfrac{y^2}{2} \right) dy - e^{-r\tau} \int_{-d_-}^{+\infty} K \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy$$

$$= x \int_{-d_-}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}\left(\sigma^2\tau - 2\sigma\sqrt{\tau}\, y + y^2 \right) \right) dy - e^{-r\tau} K \left(1 - N(-d_-) \right)$$

$$= x \int_{-d_-}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}(y - \sigma\sqrt{\tau})^2 \right) dy - e^{-r\tau} K \left(1 - N(-d_-) \right).$$

$$z = y - \sigma\sqrt{\tau}$$

$$c(t,x) = x\int_{-d_- - \sigma\sqrt{\tau}}^{+\infty} \underline{\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)} dz - e^{-r\tau}K(1 - N(-d_-)).$$

$$= x\left(1 - N(-d_- - \sigma\sqrt{\tau})\right) - e^{-r\tau}K(1 - N(-d_-)).$$

$$N(-x) = \mathbb{P}(N(0,1) \leq -x) = \mathbb{P}(-N(0,1) \geq x) = \mathbb{P}(N(0,1) \geq x) = 1 - \mathbb{P}(N(0,1) \leq x)$$

$$= 1 - N(x).$$

$$c(t,x) = x N(\underline{d_- + \sigma\sqrt{\tau}}) - e^{-r\tau} K N(d_-)$$

$$d_- + \sigma\sqrt{\tau} = \frac{\ln\left(\frac{x}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} + \sigma\sqrt{\tau} = \frac{\ln\left(\frac{x}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} + \frac{\sigma^2\tau}{\sigma\sqrt{\tau}}$$

$$= \frac{\ln\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} = d_+$$

$$\Rightarrow \quad c(t,x) = x N(d_+) - e^{-r\tau} K N(d_-).$$