Last time: Multi dim Itô

$$d f(t, X_t) = \partial_t f \, dt + \sum_{i=1}^{d} \partial_i f \, dX_t^i + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \partial_i \partial_j f \, \underbrace{d[X^i, X^j]_t}_{\text{Joint QV}}$$

Definition 9.20 (d-dimensional Brownian motion)**.** We say a d-dimensional process $W = (W^1, \dots, W^d)$ is a Brownian motion if:

(1) Each coordinate W^i is a standard 1-dimensional Brownian motion.
(2) For $i \neq j$, the processes W^i and W^j are independent.

Remark 9.21. If W is a d-dimensional Brownian motion then $\boxed{d[W^i, W^j]_t = \begin{cases} dt & i = j\,, \\ 0\,dt & i \neq j\,. \end{cases}}$

Joint QV
of ind cts mg = 0

Theorem 9.22 (Lévy). *Let M be a d-dimensional process such that:*

(1) M is a continuous martingale.

$d[M^i, M^j]$

(2) The joint quadratic variation satisfies: $d[\cancel{W}^i, \cancel{W}^j]_t = \begin{cases} dt & i = j\,, \\ 0\,dt & i \neq j\,. \end{cases}$

Then M is a d-dimensional Brownian motion.

Proof. Find $\boxed{\boldsymbol{E}_s e^{\lambda M_t^i + \mu M_t^j}}$ using Itô's formula, similar to Problem 7.5. (Recitation/Reviews) □

$\Rightarrow$ M^i & M^j are ind for $i \neq j$

& $\Rightarrow$ $M_t - M_s \sim N\left(0, \underbrace{\begin{pmatrix} t-s & & \\ & t-s & \\ & & \ddots & \\ & & & t-s \end{pmatrix}}_{(t-s)I}\right)$

Example 9.23. Let $f \in C^{1,2}$, W be a d-dimensional Brownian motion, and set $X_t = f(t, W_t)$. Find the Itô decomposition of X.

(Maybe later)

Question 9.24. *Let W be a 2-dimensional Brownian motion. Let* $X_t = \ln(|W_t|^2) = \ln((W_t^1)^2 + (W_t^2)^2)$. *Is X a martingale?*

(later)

10. Risk Neutral Pricing

$\iff C_t = C_0 \exp\left(\int_0^t R_s \, ds\right)$

Goal.

- Consider a market with a bank and one stock.
- The interest rate R_t is some adapted process.

Cash in bank $\partial_t C_t = R_t C_t$

- The stock price satisfies $dS_t = \alpha_t S_t \, dt + \sigma_t S_t \, dW_t$. (Here α, σ are adapted processes).
- Find the risk neutral measure and use it to price securities.

Definition 10.1. Let $D_t = \exp(-\int_0^t R_s \, ds)$ be the discount factor.

Remark 10.2. Note $\partial_t D = -R_t D_t$.

Remark 10.3. D_t dollars in the bank at time 0 becomes \$1 in the bank at time t.

T = maturity time

Theorem 10.4. *The (unique) risk neutral measure is given by* $\boxed{d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}}$, *where*

$$Z_T = \exp\Big(-\int_0^T \theta_t \, dW_t - \frac{1}{2} \int_0^T \theta_t^2 \, dt \Big) , \qquad \boxed{\theta_t = \frac{\alpha_t - R_t}{\sigma_t}} .$$

Theorem 10.5. *Any security can be replicated. If a security pays* V_T *at time* T, *then the arbitrage free price at time* t *is*

$$V_t = \frac{1}{D_t} \tilde{\boldsymbol{E}}_t (D_T V_T) = \tilde{\boldsymbol{E}}_t \Big(\exp\Big(\int_t^T -R_s \, ds \Big) \, V_T) \Big) .$$

Remark 10.6. We will explain the notation $d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}$ and prove both the above theorems later.

Same formula as in the Binomial model.

Definition 10.7. We say $\tilde{\boldsymbol{P}}$ is a risk neutral measure if:

(1) $\tilde{\boldsymbol{P}}$ is equivalent to $\boldsymbol{P}$ (i.e. $\tilde{\boldsymbol{P}}(A) = 0$ if and only if $\boldsymbol{P}(A) = 0$)

(2) $D_t S_t$ is a $\tilde{\boldsymbol{P}}$ martingale.

Remark 10.8. As before, if $\tilde{\boldsymbol{P}}$ is a new measure, we use $\tilde{\boldsymbol{E}}$ to denote expectations with respect to $\tilde{\boldsymbol{P}}$ and $\tilde{\boldsymbol{E}}_t$ to denote conditional expectations.

Example 10.9. Fix $T > 0$. Let Z_T be a $\mathcal{F}_T$-measurable random variable.

- Assume $\boxed{Z_T > 0}$ and $\boldsymbol{E} Z_T = 1$. $\big(Z > 0 \Rightarrow P(A) > 0 \iff \tilde{P}(A) > 0\big)$
- Define $\tilde{\boldsymbol{P}}(A) = \boldsymbol{E}(Z_T \mathbf{1}_A) = \int_A Z_T \, d\boldsymbol{P}$. $\big(E Z_T = 1 \Rightarrow \tilde{P}(\Omega) = E Z_T \mathbb{1}_\Omega = E Z_T = 1\big)$
- Can check $\tilde{\boldsymbol{E}} X = \boldsymbol{E}(Z_T X)$. That is $\int_\Omega X \, d\tilde{\boldsymbol{P}} = \int_\Omega X \, Z_T \, d\boldsymbol{P}$.
- Notation: Write $\boxed{d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}}$.

Lemma 10.10. *Let $Z_t = \boldsymbol{E}_t Z_T$. If X_t is $\mathcal{F}_t$-measurable, then* $\tilde{\boldsymbol{E}}_s X_t = \frac{1}{Z_s} \boldsymbol{E}_s(Z_t X_t)$.

Proof. You will see this in the proof of the Girsanov theorem. $\square$

Theorem 10.11 (Cameron, Martin, Girsanov). *Fix $T > 0$, and define:*

- $b_t = (b_t^1, \dots, b_t^d)$ *a d-dimensional adapted process.*
- W *a d-dimensional Brownian motion.*
- $\tilde{W}_t = W_t + \int_0^t b_s\,ds$ *(i.e. $d\tilde{W}_t = b_t\,dt + dW_t$).*
- $d\tilde{\boldsymbol{P}} = Z_T\,d\boldsymbol{P}$*, where*

$(B^i_t = \int_0^t b^i_s\, ds$

$$Z_t = \exp\Big(- \int_0^t b_s \cdot dW_s - \frac{1}{2} \int_0^t |b_s|^2\,ds \Big).$$

If Z is a martingale, then $\tilde{\boldsymbol{P}}$ is an equivalent measure under which $\tilde{W}$ is a Brownian motion up to time T.

Remark 10.12. Note $\tilde{W}_t$ is a vector.

(1) So $\tilde{W}_t = W_t + \int_0^t b_s\,ds$ means $\tilde{W}_t^i = W_t^i + \int_0^t b_s^i\,ds$, for each $i \in \{1, \dots, d\}$.
(2) Similarly, $d\tilde{W}_t = b_t\,dt + dW_t$ means $d\tilde{W}_t^i = b_t^i\,dt + d\tilde{W}_t^i$ for each $i \in \{1, \dots, d\}$.

Remark 10.13. $\int_0^t b_s \cdot dW_s$ means $\int_0^t \sum_{i=1}^d b_s^i\,dW_s^i$ (dot product).

(Sometimes $E\int_0^t Z_s |b_s|^2 ds = \infty$)

Proposition 10.14. $dZ_t = \boxed{-Z_t b_t \cdot dW_t}$. *Explicitly, in coordinates,* $dZ_t = -Z_t \sum_{i=1}^{d} b_t^i \, dW_t^i$.

Question 10.15. *Looks like Z is a martingale. Why did we assume it in Theorem 10.11?*

$$Z_t = \exp\left(-X_t - \frac{1}{2}\int_0^t |b_s|^2 \, ds\right) \qquad \left(|b_s|^2 = \sum (b_s^i)^2 \right)$$

where $X_t = + \int_0^t b_s \cdot dW_s = \sum_{i=1}^{d} \int_0^t b^i \, dW^i$

$$f(t,x) = \exp\left(-x - \frac{1}{2}\int_0^t |b_s|^2 \, ds\right)$$

$$Z_t = f(t, X_t).$$

① $\partial_t f = \exp(\quad) \cdot \left(-\frac{1}{2} |b_t|^2\right)$

② $\partial_x f = \exp(\quad)(-1)$

③ $\partial_x^2 f = \exp(\quad)(+1)$

④ $d[X, X] = \sum_{i=1}^{d} \sum_{j=1}^{d} b_t^i b_t^j \, d[W_t^i, W_t^j]$ $\left(dX = \sum b_t^i \, dW_t^i\right.$

$$= \sum_{i=1}^{d} (b_t^i)^2 \, dt = |b_t|^2 \, dt$$

$$\Rightarrow dZ = \partial_t f \, dt + \partial_x f \, dX + \frac{1}{2} \partial^2_{xx} f \, d[X,X]$$

$$= \exp(\;)\left[\left(-\frac{1}{2}|\sigma|^2 dt\right) - dX + \frac{1}{2}(1)\left(|\sigma_t|^2 dt\right)\right]$$

$$\boxed{\Rightarrow dZ_t = -Z_t \, \sigma_t \cdot dW_t} \longrightarrow M_g$$

Idea behind the proof of Theorem 10.11.

NTS $\tilde{W}$ is a BM under $\tilde{P}$.

Will show

$$\boxed{① \; [\tilde{W}, \tilde{W}]_t = [W, W]_t = t}$$ True

② $\tilde{W}$ is a $\tilde{P}$ mg ← Needs checking (Use Lemma 10.10)

① & ② + Levy ⇒ $\tilde{W}$ is a BM under $\tilde{P}$

Theorem (Theorem 10.4)**.** *The (unique) risk neutral measure is given by* $d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}$, *where*

$$Z_T = \exp\Big(-\int_0^T \theta_t \, dW_t - \frac{1}{2}\int_0^T \theta_t^2 \, dt\Big), \qquad \theta_t = \frac{\alpha_t - R_t}{\sigma_t}.$$

Proof of Theorem 10.4.

Know: $d\tilde{W} = (\cdots)\, dt + dW \to$ Girsanov gives $\tilde{P}$ under which $\tilde{W}$ is a BM. (& has formula)

~~RM~~ RNM. Want $D_t S_t$ to be a $\tilde{P}$ mg.

Compute $d(D_t S_t)$:

$$dS_t = \alpha_t S_t \, dt + \sigma_t S_t \, dW_t$$

$$dD_t = -R_t D_t \, dt$$

$$\Rightarrow d(D_t S_t) = D_t \, dS_t + S \, dD_t + \underbrace{d[S, D]}_{0}$$

$$= D_t \left(\alpha_t S_t \, dt + \sigma_t S_t \, dW \right) - R_t S_t D_t \, dt$$

$$= D_t \sigma_t S_t \left(\left(\frac{\alpha_t - R_t}{\sigma_t} \right) dt + dW_t \right)$$

$$= D_t \sigma_t S_t \left(\underline{\theta_t}\, dt + \underline{\underline{dW}} \right), \quad \boxed{\theta_t = \frac{\alpha_t - R_t}{\sigma_t}}$$

Market price of risk eq

$$= \underline{D_t \sigma_t S_t \, d\widetilde{W}_t},$$

By Girsanov $\widetilde{W}$ is a BM under $\widetilde{P}$ $\boxed{d\widetilde{W} = \theta_t\, dt + dW}$

where $d\tilde{P} = Z_T \, dP$,

& $$Z_T = \exp\left(- \int_0^T \theta_s \, dW_s - \frac{1}{2} \int_0^T \theta_s^2 \, dt \right)$$

Hence, under $\tilde{P}$,

$$d(D_t S_t) = \underbrace{\sigma_t D_t S_t \, \underbrace{d\tilde{W}}_{\text{BM under } \tilde{P}}}_{\text{Mg under } \tilde{P} \text{ !!}}$$

Theorem 10.16. *X_t represents the wealth of a self-financing portfolio if and only if $D_t X_t$ is a $\tilde{\boldsymbol{P}}$ martingale.*

Remark 10.17. The proof of the backward direction requires the *martingale representation theorem*, and is outlined on your homework.

Remark 10.18. This is the analog of Theorem 4.57 ← Same result for Binom Model.

Proof of the forward direction.

Assume X = wealth of a self fin port.

WTS: $D_t X_t$ is a $\tilde{P}$ mg.

Pf: By assumption $dX = \Delta_t \, dS_t + R_t (X_t - \Delta_t S_t) \, dt$

Self fin condition.

$$d(D_t X_t) = D_t\, dX_t + X_t\, dD_t + \underbrace{d[D, X]_t}_{0}$$

$$= -R_t D_t X_t + D_t\Big(\Delta_t\, dS + R_t (X_t - \Delta_t S_t)\, dt \Big)$$

$$= D_t \Delta_t\, dS - D_t R_t \Delta_t S\, dt \qquad \cdots (*)$$

Also note $$d(D_t S_t) = D_t\, dS + S\, dD_t + 0$$

$$= D_t\, dS - R D S\, dt$$

Then $(*) \Rightarrow$ $$d(D_t X_t) = \Delta_t \left(D_t\, dS - R_t D_t S_t\, dt \right)$$

$$= \Delta_t\, d\underbrace{(D_t S_t)}_{\tilde{P}\text{ mg!!}}$$

Here $D_t X_t$ is a $\hat{P}$ mg !!

Theorem (Theorem 10.5)**.** *Any security can be replicated. If a security pays V_T at time T, then the arbitrage free price at time t is*

$$V_t = \frac{1}{D_t}\tilde{\boldsymbol{E}}_t(D_T V_T) = \tilde{\boldsymbol{E}}_t\Big(\exp\Big(\int_t^T -R_s\,ds\Big)\,V_T\Big)\Big).$$

Remark 10.19. This is the analog of Proposition 4.1. ← Binom model

Proof of Theorem 10.5.

① Let $X_t = \frac{1}{D_t}\tilde{E}_t(D_T V_T)$

$\Rightarrow$ ⓐ $X_T = \frac{1}{D_T}\tilde{E}_T(D_T V_T) = V_T = \text{payoff}$

& (b) Compute $\tilde{E}_s(D_t X_t)$

$$= \tilde{E}_s\left(\tilde{E}_t(D_T X_T)\right)$$

$\underline{\text{tower}}$ $\tilde{E}_s(D_T X_T) = D_s X_s$

$\Rightarrow D_t X_t$ is a $\tilde{P}$ mg $\Rightarrow X =$ wealth of a self fin Port.

$\Rightarrow$ AFP at time $t = V_t = X_t = \frac{1}{D_t} \tilde{E}_t (D_T V_T)$

$$D_t = \exp\left(-\int_0^t R_s \, ds\right)$$

11. **Black Scholes Formula revisited**

- Suppose the interest rate $R_t = r$ (is constant in time).
- Suppose the price of the stock is a GBM(α, σ) (both α, σ are constant in time).

Theorem 11.1. *Consider a security that pays $V_T = g(S_T)$ at maturity time T. The arbitrage free price of this security at any time $t \leqslant T$ is given by $f(t, S_t)$, where*

$$f(t,x) = \int_{-\infty}^{\infty} e^{-r\tau} g\Big(x \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)\tau + \sigma\sqrt{\tau}\, y\Big)\Big) \frac{e^{-y^2/2}dy}{\sqrt{2\pi}}\,, \qquad \tau = T - t\,. \tag{8.4}$$

Remark 11.2. This proves Proposition 8.8.

Pf: ① $dS_t = \alpha S_t\, dt + \sigma S_t\, dW$ (GBM)

(2) Under RNM:

$$dS_t = r S_t \, dt + \sigma S_t \, d\tilde{W}$$

$$\Rightarrow S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right) t + \sigma \tilde{W}_t \right)$$

(From HW).

(4) Compute $V_t \overset{RNP}{=} \frac{1}{D_t} \tilde{E}_t\left(D_T g(S_T)\right)$

$$= \frac{1}{e^{-rt}} \tilde{E}_t\left(e^{-rT} g(S_T)\right)$$

$$= e^{-r\tau} \tilde{E}_t\, g\left(S_T\right)$$

$$\left.\begin{aligned} S_T &= S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma \tilde{W}_T\right) \\ S_t &= S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma \tilde{W}_t\right) \end{aligned}\right\} \quad S_T = S_t \exp\left(\left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\left(\tilde{W}_T - \tilde{W}_t\right)\right)$$

$$\Rightarrow V_t = e^{-r\tau} \tilde{E}_t\, g\Big(\underbrace{S_t}_{\mathcal{F}_t \text{ meas}} \exp\Big(\big(r - \tfrac{\sigma^2}{2}\big)\tau + \sigma\sqrt{\tau}\,\underbrace{\Big(\frac{\tilde{W}_T - \tilde{W}_t}{\sqrt{\tau}}\Big)}_{\text{indep of } \mathcal{F}_t}\Big)\Big)$$

$$\underset{\text{indep lemma}}{=} e^{-r\tau} \int_{y=-\infty}^{\infty} g\Big(\underbrace{S_t}_{x} \exp\Big(\big(r - \tfrac{\sigma^2}{2}\big)\tau + \sigma\sqrt{\tau}\, y\Big)\Big) \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy$$

QED.

Theorem 11.3 (Black Scholes Formula)**.** *The arbitrage free price of a European call with strike K and maturity T is given by:*

$$c(t, x) = xN(d_+(T-t, x)) - Ke^{-r(T-t)}N(d_-(T-t, x)) \tag{8.5}$$

where

$$d_\pm(\tau, x) \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big), \tag{8.6}$$

and

$$N(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy, \tag{8.7}$$

is the CDF of a standard normal variable.

Just substitute. & simplify from Thm 11.2.

Remark 11.4. This proves Corollary 8.9.