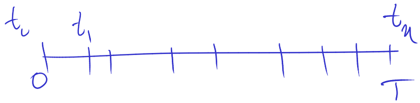
41

9. **Multi-dimensional Itô calculus**



- Let X and Y be two Itô processes.
- $P = \{0 = t_1 < t_1 \cdots < t_n = T\}$ is a partition of $[0, T]$.

$\|P\| = \max_i (t_{i+1} - t_i)$

Definition 9.1. The *joint quadratic variation* of X, Y, is defined by

$$[X,Y]_T = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}),$$

$= \lim_{\|P\| \to 0} \sum_i (\Delta_i X)(\Delta_i Y)$

Remark 9.2. The joint quadratic variation is sometimes written as $d[X,Y]_t = dX_t \, dY_t$.

QV: $[X,X]_T = \lim_{\|P\| \to 0} \sum_{i=0}^{n} (\Delta_i X)^2$

$\Delta_i X = X_{t_{i+1}} - X_{t_i}$

Lemma 9.3. $\underbrace{[X,Y]_T}_{\text{Joint QV of } X \,\&\, Y} = \frac{1}{4}(\underbrace{[X+Y,X+Y]_T}_{\text{QV of } X+Y} - \underbrace{[X-Y,X-Y]_T}_{\text{QV of } X-Y})$

Pf: $(a+b)^2 - (a-b)^2 = 4ab$

$$\Rightarrow \underbrace{\lim_{\|P\|\to 0} \sum_i \Delta_i(\cancel{X+Y})^{X}\, \Delta_i(\cancel{X-Y})^{Y}}_{[X,Y]_T} = \lim \sum_i \frac{1}{4}\underbrace{\left[\left(\Delta_i(X+Y)\right)^2 - \left(\Delta_i(X-Y)\right)^2\right]}_{\frac{1}{4}\left([X+Y,X+Y]_T - [X-Y,X-Y]_T\right)}$$

Proposition 9.4 (Product rule)**.** $d(XY)_t = X_t\, dY_t + Y_t dX_t + \boxed{d[X,Y]_t}$

If X & Y are diff,

$$\frac{d}{dt}(XY) = X\frac{dY}{dt} + \frac{dX}{dt}Y$$

$$\Rightarrow d(XY) = X\,dY + Y\,dY$$

If X, Y are stoch processes (not diff)

$$d(XY) = X\,dY + Y\,dX + d[X,Y]$$

Pf: $4XY = (X+Y)^2 - (X-Y)^2$

Itô: $d\left((X+Y)_t^2\right) = 2(X_t+Y_t)\, d(X_t+Y_t) + \frac{1}{2}\cdot 2\, d[X+Y, X+Y]$

$d\left((X-Y)_t^2\right) = 2(X_t-Y_t)\, d(X_t-Y_t) + d[X-Y, X-Y]_t$

$= 2X_t dX_t + 2Y_t dY_t - 2Y_t dX_t - 2X_t dY_t$

$+ d[X-Y, X-Y]_t.$

$$\Rightarrow d\left((X+Y)^2-(X-Y)^2\right)$$

$$= 4X_t\,dY_t + 4Y_t\,dX_t + d\left([X+Y,X+Y]_t - [X-Y,X-Y]_t\right)$$

$$\Rightarrow 4\,d(XY) = 4X\,dY + 4Y\,dX + 4\,d[X,Y] \Rightarrow \text{done!}$$

Proposition 9.5. *Say X, Y are two semi-martingales.*

- *Write $X = X_0 + B + M$, where B has bounded variation and M is a martingale.*
- *Write $Y = Y_0 + C + N$, where C has bounded variation and N is a martingale.*
- *Then $d[X,Y]_t = d[M,N]_t$.*

Remark 9.6. Recall, all processes are implicitly assumed to be *adapted* and *continuous*.

Pf: $[X,Y] = \frac{1}{4}\Big([X+Y, X+Y] - [X-Y, X-Y]\Big)$

$= \frac{1}{4}\Big([M+N, M+N] - [M-N, M-N]\Big)$ ($\because$ BV part does not change QV)

$$= \quad [M, N].$$

Corollary 9.7. *If X is a semi-martingale and B has bounded variation then* $[X, B] = 0$.

$$[X, B] = [\text{mg part of } X, \text{ mg part of } B]$$

$$= [M, 0] = 0$$

Notation.

- *d-dimensional vectors:* Write $x = (x_1, \dots, x_d) \in \mathbb{R}^d$.
- *d-dimensional* random *vectors:* $X = (X_1, \dots, X_d)$, where each X_i is a random variable.
- *d-dimensional stochastic processes:* $X_t = (X_t^1, \dots, X_t^d)$, where each X_t^i is a stochastic process.
  - ▷ For scalars (or random variables): X^i denotes the i-th power of X.
  - ▷ For vectors (or random random vectors): X^i denotes the i-th *coordinate* of X.
  - ▷ There is no ambiguity (can't take powers of vectors, or coordinates of scalars)
- Alternate notation used in many books: Use $X(t)$ for the d-dimensional stochastic process, and $X_i(t)$ for the i-th coordinate.
- May sometimes write $X = (X^1, \dots, X^d)$ for random vectors, instead of $(X_1, \dots, X_d)$.

Notation for d-dim Processes

X^{i}_{t}

$i \longrightarrow$ coordinate

$t \longrightarrow$ time

Remark 9.8 (Chain rule). If X is a differentiable function of t, then

$$d(f(t,X_t)) = \partial_t f(t,X_t)\,dt + \sum_{i=1}^{d} \partial_i f(t,X_t)\,dX_t^i$$

Remark 9.9 (Notation). $\partial_t f = \frac{\partial f}{\partial t}$, $\partial_i f = \frac{\partial f}{\partial x_i}$.

$f = f(t, x)$, $x \in \mathbb{R}^d$

$$\frac{d}{dt}\Big(f(t, X_t)\Big) \overset{\text{Chain Rule}}{=} \partial_t f \Big|_{(t,X_t)} \frac{dt}{dt} + \sum_{i=1}^{d} \partial_i f \Big|_{(t,X_t)} \frac{dX^i}{dt}$$

$$\Rightarrow \frac{d}{dt} f(t, X_t) = \partial_t f(t, X_t) + \sum_{i=1}^{d} \partial_i f(t, X_t) \frac{dX^i}{dt}$$

"Multiply by dt" & get

$$\underbrace{df(t, X_t) = \partial_t f(t, X_t)\, dt + \sum_{i=1}^{d} \partial_i f(t, X_t)\, dX^i}$$

Theorem 9.10 (Multi-dimensional Itô formula).

- *Let X be a d-dimensional Itô process.* $X_t = (X_t^1, \dots, X_t^d)$.
- *Let $f = f(t, x)$ be a function that's defined for $t \in \mathbb{R}$, $x \in \mathbb{R}^d$.*
- *Suppose $f \in C^{1,2}$. That is:*
  - ▷ *f is once differentiable in t*
  - ▷ *f is twice in each coordinate x_i* (includes $\partial_i \partial_j f$)
  - ▷ *All the above partial derivatives are continuous. Then:*

$$\underbrace{d(f(t, X_t)) = \partial_t f(t, X_t)\, dt + \sum_{i=1}^{d} \partial_i f(t, X_t)\, dX_t^i}_{\text{Chain Rule}} + \overbrace{\frac{1}{2} \sum_{i,j} \partial_i \partial_j f(t, X_t)\, d[X^i, X^j]_t}^{\text{extra!}}$$

Joint QV.

Remark 9.11 (Integral form of Itô's formula).

$$f(T, X_T) - f(0, X_0) = \int_0^T \partial_t f(t, X_t)\, dt + \sum_{i=1}^{d} \int_0^T \partial_i f(t, X_t)\, dX_t^i + \frac{1}{2} \sum_{i,j} \int_0^T \partial_i \partial_j f(t, X_t)\, d[X^i, X^j]_t$$

Remark 9.12. As with the 1D Itô, will drop the arguments (t, X_t). *Remember they are there.*

$$\int_0^T \partial_t f \, dt + \sum_{i=1}^{d} \int \partial_i f \, dX_t^i + \frac{1}{2} \sum_{i,j} \int_0^T \partial_i \partial_j f \, d[X^i, X^j]_t$$

Intuition behind Theorem 9.10.

$$P = \{0 = t_0 < t_1 \;\ldots\; t_n = T\}$$

$$f(T, X_T) - f(0, X_0) = \sum_{k=1}^{n} f(t_{k+1}, X_{t_{k+1}}) - f(t_k, X_{t_k})$$

$$\overset{\text{Taylor}}{=} \sum_{k=0}^{n-1} \underbrace{\partial_t f \,(t_{k+1} - t_k)}_{\xrightarrow{\|P\| \to 0} \int_0^T \partial_t f \, dt} + \sum_{k=0}^{n-1} \sum_{i=1}^{d} \underbrace{\partial_i f \,(X^i_{t_{k+1}} - X^i_{t_k})}_{\xrightarrow{\|P\| \to 0} \int_0^T \partial_i f \, dX^i}$$

$$+\sum_{k=0}^{n-1} \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \partial_i \partial_j f \underbrace{\left(X^i_{t_{k+1}} - X^i_{t_k}\right)\left(X^j_{t_{k+1}} - X^j_{t_k}\right)}_{\text{inc of Joint QV.}}$$

$$\xrightarrow{\|P\| \to 0} \overbrace{\frac{1}{2} \int_0^T \sum_{i=1}^{d} \sum_{j=1}^{d} \partial_i \partial_j f \, d[X^i, X^j]_t}$$

To use the d-dimensional Itô formula, we need to $\boxed{\text{compute joint quadratic variations.}}$

Proposition 9.13. *Let M, N be continuous martingales, with $\boldsymbol{E}M_t^2 < \infty$ and $\boldsymbol{E}N_t^2 < \infty$.*

(1) $MN - [M, N]$ is also a continuous martingale.

(2) Conversely if $MN - B$ is a continuous martingale for some continuous adapted, bounded variation *process B with $B_0 = 0$, then $\boxed{B = [M, N].}$*

Proof. 1 $d\big(MN - [M,N]\big) = M\,dN + N\,dM + \cancel{d[M,N]} - \cancel{d[M,N]}$

$$= \underbrace{M\,dN}_{\text{Mg}} + \underbrace{N\,dM}_{\text{Mg}}$$

(Recall: If M is a mg then $M^2 - [M,M]$ is also a mg)

Proposition 9.14. *(1)* (Symmetry) $[X,Y]=[Y,X]$

(2) (Bi-linearity) *If* $\alpha \in \mathbb{R}$, X, Y, Z *are semi-martingales,* $[X, Y+\alpha Z] = [X,Y] + \alpha[X,Z]$.

Proof.

Joint QV
$$[X, Y+\alpha Z]_T = \lim_{\|P\| \to 0} \sum (\Delta_i X)(\Delta_i (Y+\alpha Z))$$
$$= \lim_{\|P\| \to 0} \sum \Delta_i X \Delta_i Y + \alpha \sum (\Delta_i X)(\Delta_i Z)$$
$$= [X,Y]_T + \alpha [X,Z]_T$$
//

Proposition 9.15. *Let M, N be two martingales, σ, τ two adapted processes.*

- *Let* $X_t = \int_0^t \sigma_s \, dM_s$ *and* $Y_t = \int_0^t \tau_s \, dN_s$.
- *Then* $\cancel{[X, Y, X, Y]_t = \int_0^t \sigma_s \tau_s \, d[M, N]_s}$.

$$[X, Y]_t = \int_0^t \sigma_s \tau_s \, d[M, N].$$

Remark 9.16. Alternately, if $\boxed{dX_t = \sigma_t \, dM_t}$ and $\boxed{dY_t = \tau_t \, dN_t}$, then $d[X, Y]_t = \sigma_t \tau_t \, d[M, N]_t$.

Intuition.

Recall $[X, X]_t = \int_0^t \sigma_s^2 \, d[M, M]_s$

$$X_T = \lim_{\|P\| \to 0} \sum \sigma_{t_i} \Delta_i M \qquad ? \qquad \Delta_i X = X_{t_{i+1}} - X_{t_i}$$

$$Y_T = \lim_{\|P\| \to 0} \sum \tau_{t_i} \Delta_i N \quad) \qquad \approx \sigma_{t_i} \left(M_{t_{i+1}} - M_{t_i} \right)$$

$$\Delta_i Y \approx \tau_{t_i} \left(N_{t_{i+1}} - N_{t_i} \right)$$

$$[X, Y] \approx \sum (\Delta_i X)(\Delta_i Y)$$

$$= \sum (\sigma_i \Delta_i M)(\tau_i \Delta_i N) \longrightarrow \int_0^T \sigma_t \tau_t \, d[M, N]_t$$

Proposition 9.17. *If M, N are continuous martingales, $\boldsymbol{E}M_t^2 < \infty$, $\boldsymbol{E}N_t^2 < \infty$ and M, N are independent, then $[M, N] = 0$.*

Remark 9.18 (Warning)*.* Independence implies $\boldsymbol{E}(M_tN_t) = \boldsymbol{E}M_t\boldsymbol{E}N_t$. But it *does not* imply $\boldsymbol{E}_s(M_tN_t) = \boldsymbol{E}_sM_t\boldsymbol{E}_sN_t$. So you can't use this to show MN is a martingale, and hence conclude $[M, N] = 0$.

Correct proof.

WRONG (Not fixable!).

$$\boxed{E_s(M_tN_t) \overset{\text{indep}}{=} E_sM_t\, E_sN_t} \leftarrow \text{FALSE}$$

$$= M_s N_s$$

$\Rightarrow M_tN_t$ is a mg $\Rightarrow [M,N] = 0$

Correct Pf: Claim $E[M,N]_T^2 = 0$

$$E[M,N]_T^2 \approx E\left(\sum (\Delta_i M)(\Delta_i N)\right)^2$$

$$= E\left[\sum_{i,j} (\Delta_i M)(\Delta_i N)\,(\Delta_j M)(\Delta_j N)\right]$$

$$= E\sum_{i=0}^{n-1} (\Delta_i M)^2 (\Delta_i N)^2 + 2E\sum_{j=1}^{n-1}\sum_{i=0}^{j-1} (\Delta_i M)(\Delta_j M)(\Delta_i N)(\Delta_j N)$$

$$= \quad \| \quad + 2 \sum_{j=1}^{n-1} \sum_{i=0}^{j-1} E(\Delta_i M \, \Delta_j M) \; E((\Delta_i N) \, \Delta_j N)$$

$$= \quad \| \quad + 2 \sum \sum E \, \underbrace{E_{t_{i+1}} (\Delta_i M \, \Delta_j M)}_{0\,!} \cdot E(\quad)$$

$$\because \; E_{t_{i+1}}(\Delta_j M) = E_{t_{i+1}}(M_{t_{j+1}} - M_{t_j}) = M_{t_{i+1}} - M_{t_{i+1}} = 0$$

$$= E \sum_{i=0}^{n-1} (\Delta_i M)^2 (\Delta_i N)^2 \quad + \quad 0$$

$$= \sum_{i=0}^{n-1} E(\Delta_i M)^2 \; E(\Delta_i N)^2.$$

$$\leq \underbrace{\max_i E(\Delta_i M)^2}_{\substack{\downarrow \ (\text{since } M \text{ is cts!}) \\ 0}} \cdot \underbrace{\sum_i E(\Delta_i N)^2}_{\substack{\downarrow \\ E[N,N]_T \ < \infty}} = 0$$

Remark 9.19. $[M, N] = 0$ does not imply M, N are independent. For example:

- ~~Let $M_t = \int_0^t \mathbf{1} W_s < 0 \, dW_s$~~
- ~~Let $N_t = \int_0^t \mathbf{1} W_s \geqslant 0 \, dW_s$~~

$$\left.\begin{array}{l} M_t = \int_0^t \mathbb{1}_{\{W_s < 0\}} \, dW_s \\ N_t = \int_0^t \mathbb{1}_{\{W_s \geqslant 0\}} \, dW_s \end{array}\right\} \begin{array}{l} d[M,N] = \mathbb{1}_{\{W_s<0\}} \mathbb{1}_{\{W_s \geqslant 0\}} \, ds \\ \quad = 0 \, ds \end{array}$$

But $M_t + N_t = \int_0^t 1 \, dW_s = W_t \leftarrow \Rightarrow M$ & N are Not ind.

Definition 9.20 (d-dimensional Brownian motion)**.** We say a d-dimensional process $W = (W^1, \dots, W^d)$ is a Brownian motion if:

(1) Each coordinate W^i is a standard 1-dimensional Brownian motion.
(2) For $i \neq j$, the processes W^i and W^j are independent.

Remark 9.21. If W is a d-dimensional Brownian motion then
$$\boxed{d[W^i, W^j]_t = \begin{cases} dt & i = j\,, \\ 0\,dt & i \neq j\,. \end{cases}}$$

Theorem 9.22 (Lévy). *Let M be a d-dimensional process such that:*

(1) M is a continuous martingale.

(2) The joint quadratic variation satisfies: $d[W^i, W^j]_t = \begin{cases} dt & i = j\,, \\ 0\,dt & i \neq j\,. \end{cases}$

Then M is a d-dimensional Brownian motion.

Proof. Find $\boldsymbol{E}_s e^{\lambda M_t^i + \mu M_t^j}$ using Itô's formula, similar to Problem 7.5. $\square$

Example 9.23. Let $f \in C^{1,2}$, W be a d-dimensional Brownian motion, and set $X_t = f(t, W_t)$. Find the Itô decomposition of X.

Question 9.24. *Let W be a 2-dimensional Brownian motion. Let $X_t = \ln(|W_t|^2) = \ln((W_t^1)^2 + (W_t^2)^2)$. Is X a martingale?*

10. **Risk Neutral Pricing**

Goal.

- Consider a market with a bank and one stock.
- The interest rate R_t is some adapted process.
- The stock price satisfies $dS_t = \alpha_t S_t \, dt + \sigma_t S_t \, dW_t$. (Here α, σ are adapted processes).
- Find the risk neutral measure and use it to price securities.

Definition 10.1. Let $D_t = \exp(-\int_0^t R_s \, ds)$ be the discount factor.

Remark 10.2. Note $\partial_t D = -R_t D_t$.

Remark 10.3. D_t dollars in the bank at time 0 becomes \$1 in the bank at time t.

Theorem 10.4. *The (unique) risk neutral measure is given by $d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}$, where*

$$Z_T = \exp\Big(-\int_0^T \theta_t \, dW_t - \frac{1}{2}\int_0^T \theta_t^2 \, dt\Big), \qquad \theta_t = \frac{\alpha_t - R_t}{\sigma_t}.$$

Theorem 10.5. *Any security can be replicated. If a security pays V_T at time T, then the arbitrage free price at time t is*

$$V_t = \frac{1}{D_t} \tilde{\boldsymbol{E}}_t(D_T V_T).$$

Remark 10.6. We will explain the notation $d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}$ and prove both the above theorems later.

Definition 10.7. We say $\tilde{\boldsymbol{P}}$ is a risk neutral measure if:

(1) $\tilde{\boldsymbol{P}}$ is equivalent to $\boldsymbol{P}$ (i.e. $\tilde{\boldsymbol{P}}(A) = 0$ if and only if $\boldsymbol{P}(A) = 0$)
(2) $D_t S_t$ is a $\tilde{\boldsymbol{P}}$ martingale.

Remark 10.8. As before, if $\tilde{\boldsymbol{P}}$ is a new measure, we use $\tilde{\boldsymbol{E}}$ to denote expectations with respect to $\tilde{\boldsymbol{P}}$ and $\tilde{\boldsymbol{E}}_t$ to denote conditional expectations.

Example 10.9. Fix $T > 0$. Let Z_T be a $\mathcal{F}_T$-measurable random variable.

- Assume $Z_T > 0$ and $\boldsymbol{E} Z_T = 1$.
- Define $\tilde{\boldsymbol{P}}(A) = \boldsymbol{E}(Z_T \mathbf{1}_A) = \displaystyle\int_A Z_T \, d\boldsymbol{P}$.
- Can check $\tilde{\boldsymbol{E}} X = \boldsymbol{E}(Z_T X)$. That is $\displaystyle\int_\Omega X \, d\tilde{\boldsymbol{P}} = \int_\Omega X \, Z_T \, d\boldsymbol{P}$.
- Notation: Write $d\tilde{\boldsymbol{P}} = Z_T \, d\boldsymbol{P}$.

Lemma 10.10. *Let $Z_t = \boldsymbol{E}_t Z_T$. If X_t is $\mathcal{F}_t$-measurable, then $\tilde{\boldsymbol{E}}_s X = \frac{1}{Z_s} \tilde{\boldsymbol{E}}_s(Z_t X_t)$.*

Proof. You will see this in the proof of the Girsanov theorem in part 2 of this course. $\square$

Theorem 10.11 (Cameron, Martin, Girsanov)**.** *Fix $T > 0$, and define:*

- $b_t = (b^1_t, \dots, b^d_t)$ *a d-dimensional adapted process.*
- W *a d-dimensional Brownian motion.*
- $\tilde{W}_t = W_t + \int_0^t b_s\, ds$ *(i.e. $d\tilde{W}_t = b_t\, dt + d\tilde{W}_t$).*
- $d\tilde{\boldsymbol{P}} = Z_T\, d\boldsymbol{P}$*, where*

$$Z_t = \exp\left(- \int_0^t b_s \cdot dW_s - \frac{1}{2} \int_0^t |b_s|^2\, ds \right).$$

If Z is a martingale, then $\tilde{\boldsymbol{P}}$ is an equivalent measure under which $\tilde{W}$ is a Brownian motion up to time T.