Last time: Market $\to \begin{cases} \text{M.M} \to (\text{interest rate } r) \\ \text{Stock} \to \text{GBM}(\alpha, \sigma): \quad dS = \alpha S\,dt + \sigma S\,dW \end{cases}$ $\quad C_t = C_0 e^{rt} \quad (\partial_t C = rC)$

Security with payoff $V_T = g(S_T)$ at time T

B.S.M. PDE: $\quad \partial_t f + r x \partial_x f + \frac{\sigma^2}{2} x^2 \partial_x^2 f = r f$

T.C.: $\quad f(T, x) = g(x)$

& B.C.

Last time: ① If $X_t = f(t, S_t)$ is the wealth of the rep portfolio, then f solves the B.S.M PDE (with BC & T.C. $f(T, x) = g(x)$)

② Conversely if f solves the $\overbrace{\text{BSM PDE}}$ (& B.C. & T.C) Then the security can be replicated & $X_t = f(t, S_t)$ is the wealth of the R. port.

Proof of Theorem 8.4. Last time:

Choose $X_0 = f(0, S_0)$

Chose $\Delta_t = \partial_x f(t, S_t)$

} Let X_t = wealth of a self fin Port with initial capital X_0 & holds Δ_t shares of stock at time t.

Set $Y_t = \underline{\underline{e}}^{-rt} X_t$. (Recall $dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t)dt$)

Compute dY_t = last time = $d\left(e^{-rt} f(t, S_t)\right)$

$$\Rightarrow d\left(Y_t - e^{-rt} f(t, S_t)\right) = 0$$

$$\Rightarrow Y_t - e^{-rt} f(t, S_t) - \left(Y_0 - f(0, S_0)\right) = \int_0^t 0\, ds + \int_0^t 0\, dW = 0$$

$$\Rightarrow e^{-rt} X_t - e^{-rt} f(t, S_t) = \underline{X_0} - f(0, S_0) = 0 \text{ (by choice of } X_0)$$

$$\Rightarrow X_t = f(t, S_t)$$

$$\Rightarrow \underline{\underline{X_T}} = f(T, S_T) = g(S_T) = \underline{\underline{V_T}} = \text{payoff of security}$$

$\Rightarrow X$ = wealth of the Rep port.

$$\Rightarrow f(t, S_t) =$$

QED.

Proof of Theorem 8.4 (without discounting).

Start with ① $X_0 = f(0, S_0)$

② Choose $\underline{\Delta_t = \partial_x f(t, S_t)}$ (Delta Hedging)

Want to show: X is a rep port & $X_t = f(t, S_t)$

① By def of self fin: $dX_t = \Delta_t \, dS + r(X_t - \Delta_t S_t)\,dt$

$\Rightarrow dX_t = \Delta_t(\alpha S\,dt + \sigma S\,dW) + r(X_t - \Delta_t S_t)\,dt$

$$(*) \quad \Rightarrow dX_t = \sigma S \overset{\partial_x f(t, S_t)}{\Delta_t} \, dW_t + \left(r X_t + (\alpha - r) \Delta_t S_t \right) dt$$

(2) By Itô:

$$d \overbrace{f(t, S_t)}^{Y_t} = \partial_t f \, dt + \partial_x f \, dS + \frac{1}{2} \partial_x^2 f \, d[S, S]$$

$$= \partial_t f \, dt + \partial_x f \left(\alpha S \, dt + \sigma S \, dW \right) + \frac{1}{2} \partial_x^2 f \, S^2 \sigma^2 \, dt$$

$$\Rightarrow dY_t = \underbrace{\left(\partial_t f + \alpha S \partial_x f + \frac{\sigma^2 S^2}{2} \partial_x^2 f \right)} dt + \underbrace{\partial_x f}_{\Delta_t} \sigma S \, dW$$

$$\boxed{\Rightarrow dY_t = \big((\alpha - r) S \partial_x f + r f\big) dt + \Delta_t \sigma S \, dW} \quad (**)$$

$$\left(\text{Using } \partial_t f + r x \partial_x f + \frac{\sigma^2}{2} x^2 \partial_x^2 f = r f \right)$$

$$\Rightarrow d(X_t - Y_t) = r\big(X_t - \underbrace{f(t, S_t)}_{Y}\big) dt + 0 \, dW$$

$$\Rightarrow d(X_t - Y_t) = r(X_t - Y_t) \, dt$$

$$\Rightarrow \quad \partial_t (X_t - Y_t) = r(X_t - Y_t)$$

$$\left.\begin{aligned}\Rightarrow X_t - Y_t &= (X_0 - Y_0)\cdot e^{rt} \\ &= \big(X_0 - f(0,S_0)\big)\, e^{rt} = 0\end{aligned}\right\} \Rightarrow X_t = Y_t = f(t, S_t).$$

$$\Rightarrow X_T = Y_T = f(T, S_T) = g(S_T) = V_T$$

$\Rightarrow$ X is the wealth of the Rep Port.

& $X_t = f(t, S_t)$

Q.E.D.

Remark 8.12. The arbitrage free price does not depend on the mean return rate!

GBM:
$$dS_t = \underbrace{\alpha}_{\text{Mean return rate.}} S\,dt + \sigma S\,dW$$

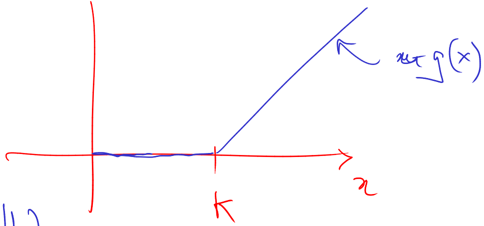
Question 8.13. *Consider a European call with maturity T and strike K. The payoff is $V_T = (S_T - K)^+$. Our proof shows that the arbitrage free price at time $t \leqslant T$ is given by $V_t = c(t, S_t)$, where c is defined by (8.5). The proof uses Itô's formula, which requires c to be twice differentiable in x; but this is clearly false at $t = T$. Is the proof still correct?*

$V_T = (S_T - K)^+ = g(S_T)$, where $g(x) = (x - K)^+$

Q: Is g diff (NO)

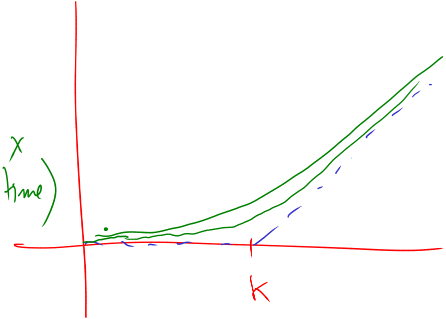
f solves BSM PDE

T.C. $f(T, x) = g(x) = (x - K)^+$ (not diff)



$f(t,x)$ for $t < T$

for $t < T$: f is diff (twice in x
one in time)



can apply Itô.

My proof will show $X_t = f(t, S_t)$ for all $\underline{t < T}$

take $\lim_{t \to T}$ & get $X_T = f(T, S_T)$.

Proposition 8.14 (Put call parity). *Consider a European put and European call with the same strike K and maturity T.*

- ▷ $c(t, S_t) =$ *AFP of call (given by (8.5))*
- ▷ $p(t, S_t) =$ *AFP of put.*

Then $c(t,x) - p(t,x) = x - Ke^{-r(T-t)}$, *and hence* $p(t,x) = Ke^{-r(T-t)} - x - c(t,x)$.

Know

$$c(t,x) = x\,N(d_+) - K e^{-r(T-t)} N(d_-)$$

$$d_\pm = \frac{1}{\sigma\sqrt{t}}\left(\ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right)$$

8.3. **The Greeks.** Let $c(t,x)$ be the arbitrage free price of a European call with maturity T and strike K when the spot price is x. Recall

$$c(t,x) = xN(d_+) - Ke^{-r\tau}N(d_-)\,, \quad d_\pm \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big)\,, \quad \tau = T - t\,.$$

Definition 8.15. The *delta* is $\partial_x c$.

Remark 8.16 (Delta hedging rule). $\boxed{\Delta_t = \partial_x c(t, S_t).}$

Proposition 8.17. $\partial_x c = N(d_+)$

$$\partial_x c = \partial_x \Big(x N(d_+) - K e^{-r\tau} N(d_-) \Big)$$

$$= N(d_+) + x N'(d_+) \cdot d_+' - K e^{-r\tau} N'(d_-) d_-'$$

① $$d_{\pm}' = \partial_x \left(\frac{1}{\sigma\sqrt{\tau}} \left(\ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau \right) \right)$$

$$= \frac{1}{x\sigma\sqrt{\tau}}$$

② $$N(x) = \int_{-\infty}^{x} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}} \Rightarrow N'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

③ $d_+^2 - d_-^2$:

$$d_{\pm} = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{x}{K}\right) + r\tau \pm \frac{\sigma^2\tau}{2}\right)$$

$$\Rightarrow d_+^2 - d_-^2 = 4\frac{1}{\sigma^2\tau}\left(\ln\left(\frac{x}{K}\right) + r\tau\right)\left(\frac{\sigma^2\tau}{2}\right)$$

$$= 2\left(\ln\left(\frac{x}{K}\right) + r\tau\right)$$

$$\Rightarrow e^{-d_-^2/2} = e^{-d_+^2/2 + \ln\left(\frac{x}{K}\right) + r\tau} = e^{-d_+^2/2}\,\frac{x}{K}\,e^{r\tau}$$

Hence $\partial_x c = N(d_+) + x N'(d_+) \cdot d'_+ - K e^{-r\tau} N'(d_-) d'_-$

$$= N(d_+) + d'_+ \left[x \frac{e^{-d_+^2/2}}{\sqrt{2\pi}} - K e^{-r\tau} \frac{e^{-d_-^2/2}}{\sqrt{2\pi}} \right]$$

$$= N(d_+) + \frac{d'}{\sqrt{2\pi}} \left(x e^{-d_+^2/2} - K e^{-r\tau} e^{-d_+^2/2} \frac{x}{K} e^{+r\tau} \right)$$

$$= N(d_+)$$

Definition 8.18. The *Gamma* is $\partial_x^2 c$ and is given by $\partial_x^2 c = \dfrac{1}{x\sigma\sqrt{2\pi\tau}} \exp\left(\dfrac{-d_+^2}{2}\right)$.

Definition 8.19. The *Theta* is $\partial_t c$, and is given by $\partial_t c = -rKe^{-r\tau}N(d_-) - \dfrac{\sigma x}{2\sqrt{\tau}} N'(d_+)$

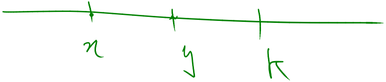
$$\partial_x^2 c = \partial_x \, \partial_x c = \partial_x \left(N(d_+) \right) = N'(d_+) \cdot d_+'$$

Proposition 8.20. *(1) c is strictly increasing as a function of x.*
(2) c is convex as a function of x.
(3) c is decreasing as a function of t.

(1) c inc as a fn of x means $c(t, y) > c(t, x)$ whenever $y > x$

Pf: $\partial_x c > 0 \Rightarrow c$ is inc as a fn of x

$$\partial_x c = N(d_+) > 0$$

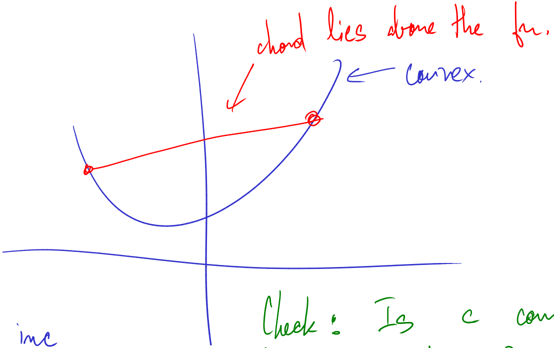


(2) Convex :

↓

A fn is convex if the derivative is inc

(i.e. the second derivative > 0)



<u>Check:</u> Is c convex as a fn of x ?

i.e. Is $\partial_x^2 c > 0$?

$$\partial_x^2 c = \text{Gamma} = \frac{1}{x\sigma\sqrt{2\pi T}} e^{-d_+^2/2} > 0$$ ✓

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(3) c is decreasing as a fn of time

(Check: $\partial_t c = \text{Theta} = \text{formula} < 0$)

Remark 8.21. To properly hedge a short call, you always borrow from the bank. Moreover $\Delta_T = 1$ if $S_T > K$, $\Delta_T = 0$ if $S_T < K$.

Delta Hedging: $\Delta_t =$ # shares in Rep port at time t

$$= \partial_x c(t, S_t)$$

$\Rightarrow$ Cash Balance: $c(t, S_t) - \partial_x c(t, S_t)\, S_t$ $\quad (\tau = T - t)$

Put $x = S_t$: $c(t,x) - x\,\partial_x c(t,x) = \cancel{x N(d_+)} - K e^{-r\tau} N(d_-) - \cancel{x N(d_+)}$

$$= -K e^{-r\tau} N(d_-) < 0$$

$$\Delta_t \xrightarrow{t \to T} \begin{cases} 1 & S_T > K \\ 0 & S_T < K \end{cases}$$

(Compute $\overbrace{\lim_{t \to T} \partial_x c(t, x)}$ $= \begin{cases} 1 & x > K \\ 0 & x < K \\ 1/2. & x = K \end{cases}$

You check.

Remark 8.22 (Delta neutral, Long Gamma). Say x_0 is the spot price at time t.

- Short $\partial_x c(t, x_0)$ shares, and buy one call option valued at $c(t, x_0)$.
- Put $M = x_0 \partial_x c(t, x_0) - c(t, x_0)$ in the bank.
- What is the portfolio value when if the stock price is x (and we hold our position)?
  - ▷ *(Delta neutral)* Portfolio value $= c(t, x)$ – tangent line.
  - ▷ *(Long gamma)* By convexity, portfolio value is always non-negative.

$x_0 =$ Spot price of stock

Portfolio - $\begin{cases} -\partial_x c(t, x_0) \text{ shares} \\ 1 \text{ call option.} \end{cases}$

Portfolio value if spot price is x

$$= c(t,x) - x\,\partial_x c(t,x_0) + M$$

$$= c(t,x) - x\,\partial_x c(t,x_0) + x_0\,\partial_x c(t,x_0) - c(t,x_0)$$

$$= c(t,x) - \Big[\underbrace{c(t,x_0) + (x-x_0)\,\partial_x c(t,x_0)}_{\text{tangent line to } c(t,x) \text{ at } x_0}\Big]$$

Convex functions are always above the tangent line