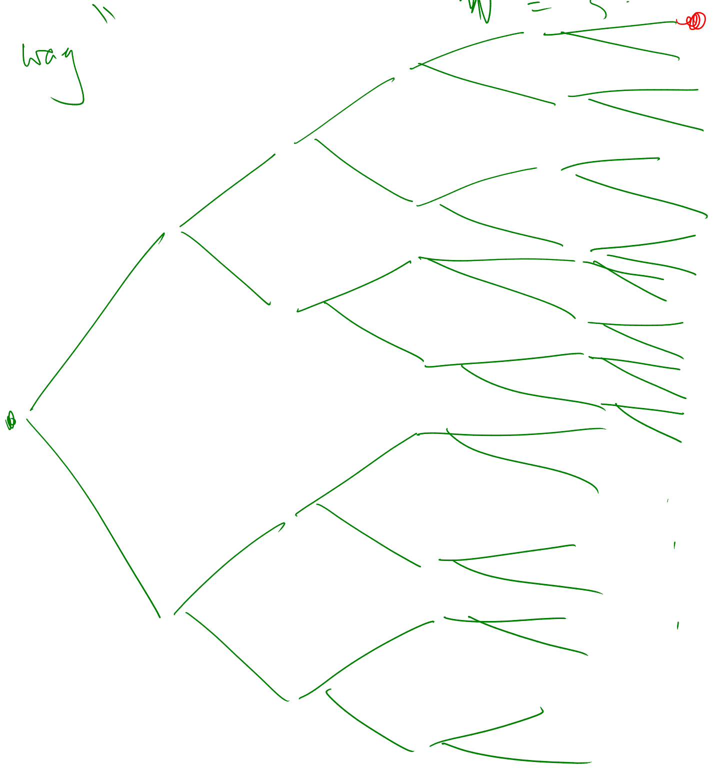
Q3 $\longrightarrow$ Security pays S_N^3 at time $N=5$

$$\left.\begin{aligned} u &= 1.1 \\ d &= 0.95 \\ r &= 0.05 \end{aligned}\right\} \quad \begin{aligned} \tilde{p} &= \frac{1+r-d}{u-d} = \frac{2}{3} \\ \tilde{q} &= \frac{1}{3} \end{aligned}$$

Neutral formula: $V_n = \frac{1}{D_n} \underbrace{\tilde{E}_n\left(D_N S_N^3\right)}$ (Want $n=1$)

"Direct way"

$N = 5.$



$S_5^3 = u^5 S_0 =$

$\rightarrow$ 32 #'s

$V_4 \rightarrow$ 16 #'s

$V_3 \rightarrow$ 8 #'s

Shorter Way: HW 2: If $V_N = g(S_N)$

then f for all $n \le N$, knows $V_n = f_n(S_n)$

& $$\boxed{f_n(s) = \frac{1}{1+r}\left(\tilde{p}\, f_{n+1}(us) + \tilde{q}\, f_{n+1}(ds)\right).}$$

(Reason: $f_N(s) = g(s)$

$\underline{n = N-1}$: knows $V_n = \frac{1}{1+r}\tilde{E}_n V_{n+1} = \frac{1}{1+r}\tilde{E}_n f_{n+1}(S_{n+1})$

$$= \frac{1}{1+r}\left(\tilde{E}_n f_{n+1}\left(S_n \cdot X_{n+1}\right)\right)$$

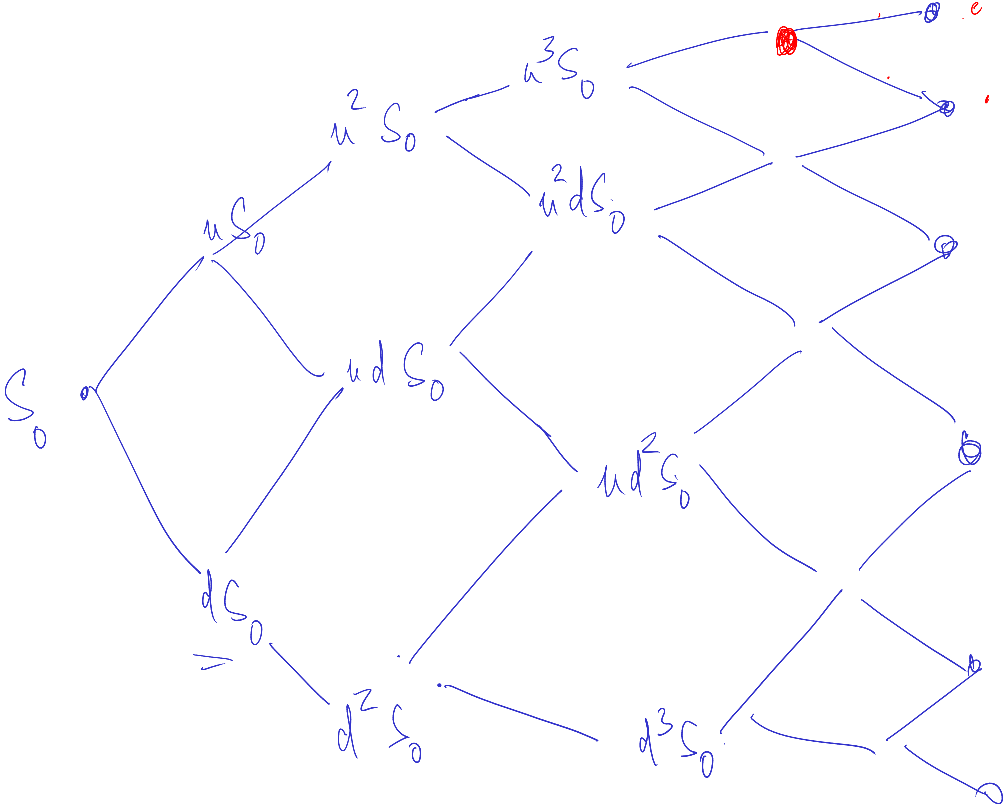
$$\left(X_{n+1} = \begin{cases} u & \omega_{n+1} = 1 \\ d & \omega_{n+1} = -1 \end{cases}\right)$$

$$= \underbrace{\frac{1}{1+r}\left(\tilde{p}\, f_{n+1}(u S_n) + \tilde{q}\, f_{n+1}(d S_n)\right)}$$

Exam Q: Expect $V_n = f_n(S_n)$ & $\boxed{f_n = \frac{1}{1+r}\left(f_{n+1}(us) + f_{n+1}(ds)\right)}$

Draw tree of stock prices (not coin tosses)

$$f_n(s) \quad \frac{1}{1+r}\left(f_{n+1}(us)\tilde{p} + f_{n+1}(ds)\tilde{q} \right)$$



$V_5 = (u^5 S_0)^3$

$(u^4 d S_0)^3$

Shorter trick:

$$\tilde{E}_n S_{n+1}^3 = \tilde{E}_n \left(S_n X_{n+1}\right)^3 \quad \text{when} \quad X_{n+1} = \begin{cases} u & \omega_{n+1} = 1 \\ d & \omega_{n+1} = -1 \end{cases}$$

$$= \tilde{E}_n \left(S_n^3 X_{n+1}^3\right)$$

$$= S_n^3 \tilde{E}_n X_{n+1}^3 = S_n^3 \tilde{E} X_{n+1}^3$$

$$= S_n^3 \left(\tilde{p} u^3 + \tilde{q} d^3\right)$$

$$\Rightarrow \quad \tilde{E}_n S^3_{n+1} = \underline{\underline{S^3_n}} \underbrace{\left(\tilde{p} u^3 + \tilde{q} d^3\right)}$$

$$\Rightarrow \quad \tilde{E}_n S^3_{n+2} \overset{\text{tower.}}{=\!=} S^3_n \left(\tilde{p} u^3 + \tilde{q} d^3\right)^2$$

$$\Rightarrow \quad \tilde{E}_1 S^3_5 = S^3_1 \left(\tilde{p} u^3 + \tilde{q} d^3\right)^4$$

$$\Rightarrow \quad V_1 = \frac{1}{(1+r)^4} \cdot \tilde{E}_1 S^3_5 \to \underline{\text{Le}} \quad \frac{S^3_1 \left(\tilde{p} u^3 + \tilde{q} d^3\right)^4}{(1+r)^4}$$

(Plug in #'s & solve)

Q4: Compute $E\left(\underbrace{\int_0^t e^{-2s}\,dW_s}\right)^4 \longrightarrow$ Idea: Write $\left(\int_0^t e^{-2s}\,dW_s\right)^4 = \int_0^t b_s\,ds + \int_0^t \sigma_s\,dW_s$

$E = 0$

Choose $X_t = \int_0^t e^{-2s}\,dW_s$ & $f(x) = x^4$

$\partial_x f = 4x^3$, $\partial_x^2 f = 12x^2$, $\partial_t f = 0$, $d[X,X]_t = e^{-4t}\,dt$

$$dX = \underbrace{e^{-2s}}_{\text{square}}\ \underbrace{dW_s}_{ds}$$

$$d[X,X] = e^{-4s}\,ds$$

$$\Rightarrow d(X^4) = \partial_t f \, dt + \partial_x f \, dX + \frac{1}{2} \partial_x^2 f \, d[X,X]$$

$$= 0 + 4X^3 e^{-2t} dW_t + \frac{1}{2} \cdot 12 X_t^2 \, e^{-4t} dt$$

$$= 4X^3 e^{-2t} dW_t + 6X_t^2 e^{-4t} dt$$

$$\Rightarrow X_t^4 = \left(\int_0^t e^{-2s} dW_s \right)^4 = \underbrace{4 \int_0^t X_s^3 e^{-2s} dW_s}_{} + 6 \int_0^t X_s^2 e^{-4s} ds$$

$$\Rightarrow E X_t^4 = 0 + 6 E \int_0^t X_s^2 e^{-4s} \, ds$$

$$= 6 \int_0^t E X_s^2 \, ds$$

$$E X_s^2 = E\left(\int_0^s e^{-2r} \, dW_r \right)^2 = \int_0^s e^{-4r} \, dr$$

Shorter Sol to Q4:

$$\int_0^t \underbrace{e^{-2s}} dW_s \longrightarrow \lim_{\|P\|\to 0} \sum \underbrace{e^{-2t_i}\left(W_{t_{i+1}} - W_{t_i}\right)}_{\text{Normal!}}$$

is normally dist

$$E\int_0^t e^{-2s}\, dW_s = 0$$

$$E\left(\int_0^t e^{-2s}\, dW_s\right)^2 = \underbrace{\int_0^t e^{-4s}\, ds}_{A}$$

$$\Rightarrow E\left(\int_0^t e^{-2s}\, dW_s\right)^4 = 3\left[\underbrace{E\left(\int_0^t e^{-2s}\, dW_s\right)^2}_{A}\right]^2$$

Q5) $M_t = \underbrace{\int_0^t s W_s \, ds}$ Find $E\left(M_t^2 - [M,M]_t\right)$

($M^2 - [M,M]$ is a mg $\Rightarrow E(\quad) = 0$

Won't work because $\longrightarrow$ M is not a mg)

$E\left(M^2 - [M,M]\right) \longrightarrow$ ① $[M,M]_t = 0$ (Riemann Int!!)

$\|$

$$E M_t^2 = E\left(\int_0^t s W_s \, ds\right)^2 = E\left(\int_0^t s W_s \, ds\right)\left(\int_0^t r W_r \, dr\right)$$

$$= \mathbb{E}\left[\int_{s=0}^{t}\int_{r=0}^{t} s\, r\, W_s W_r \, dr\, ds\right]$$

$$= \int_{s=0}^{t}\int_{r=0}^{t} s\, r\, \underbrace{\mathbb{E}(W_s W_r)}_{(s \wedge r)} \, dr\, ds$$

& integrate!

Alternate approach :

$$M_t = \int_0^t s W_s \, ds \longrightarrow dM_t = t W_t \, dt$$

Choose $f(t,x) = \dfrac{t^2 x}{2}$

$$\Rightarrow d\left(\frac{t^2 W_t}{2}\right) = t W_t \, dt + \frac{t^2}{2} dW_t + 0$$

$$\Rightarrow \frac{t^2 W_t}{2} = \underbrace{\int_0^t s W_s \, ds}_{M_t} + \int_0^t \frac{s^2}{2} \, dW_s$$

$$\Rightarrow M_t = \frac{t^2 W_t}{2} - \int_0^t \frac{s^2}{2} \, dW_s$$

$$\Rightarrow E M_t^2 = E \left(\qquad \right)^2 = E\left(\frac{t^4 W_t^2}{4} + \left(\int_0^t \frac{s^2}{2} \, dW_s \right)^2 + t^2 W_t \int_0^t \frac{s^2}{2} \, dW_s \right)$$

$$= \frac{t^4 t}{4} + \int_0^t \frac{s^4}{4} ds + t^2 E W_t \int_0^t \frac{s^2}{2} dW_s$$

Common mistake:

$$E\left(W_t \int_0^t \frac{s^2}{2} dW_s\right) = E \int_0^t \left(W_t \frac{s^2}{2}\right) dW_s$$

$$= 0$$

Wrong because $\frac{W_t s^2}{2}$ is not $\mathcal{F}_s$ meas. $\rightarrow$ Itô int not defined!!

Q: Find a way to compute $E\left(W_t \int_0^t \frac{s^2}{2}\, dW_s\right)$.

Use $W_t \int_0^t \frac{s^2}{2}\, dW_s = W_t \left(\frac{t^2 W_t}{2} - \int_0^t s\, W_s\, ds \right)$