8. Black Scholes Merton equation

- Cash: simple interest rate r in a bank.
- Let $\boxed{\Delta t}$ be small. $C_{n\,\Delta t}$ be cash in bank at time $n\,\Delta t$.
- Withdraw at time $n\,\Delta t$ and immediately re-deposit: $C_{(n+1)\Delta t} = (1 + r\,\Delta t)C_{n\Delta t}$.
- Set $t = n\Delta t$, send $\Delta t \to 0$: $\boxed{\partial_t C = rC}$ and $C_t = C_0 e^{rt}$.
- r is called the continuously compounded interest rate.
- Alternately: If a bank pays interest rate ρ after time T, then the equivalent continuously compounded interest rate is $r = \frac{1}{T}\ln(1 + \rho)$.

$(n \in \mathbb{N})$

$$C_{(n+1)\Delta t} - C_{n\Delta t} = r\,\Delta t\, C_{n\Delta t}$$

$$\Rightarrow \frac{C_{t+\Delta t} - C_t}{\Delta t} = r\,C_t \quad : \text{Send } \Delta t \to 0$$

$$\partial_t C_t$$

($\alpha \to$ Mean return rate)

- Stock price: $S_{t+\Delta t} = (1 + \cancel{r}\,\alpha\,\Delta t)S_t + \boxed{\text{noise}}$.
  - ▷ Variance of noise should be proportional to Δt.
  - ▷ Variance of noise should be proportional to S_t.
- $S_{t+\Delta t} - S_t = \cancel{r}\,\alpha\, S_t\,\Delta t + \sigma S_t(\Delta W_t)$.

Definition 8.1. A $\boxed{\text{Geometric Brownian motion}}$ with parameters α, σ is defined by:

$$dS_t = \alpha S_t\,dt + \sigma S_t \boxed{dW_t}.$$

Model for Stock price.

- α: Mean return rate (or percentage drift)
- σ: volatility (or percentage volatility)

Proposition 8.2. $\boxed{S_t = S_0 \exp\Big(\Big(\alpha - \frac{\sigma^2}{2}\Big)t + \sigma W_t\Big)}$

Set $Y = \ln S_t$ & Itô:

$\Rightarrow dY = \partial_t f \, dt + \partial_x f \, dS + \frac{1}{2}\partial_x^2 \, d[S,S]$

$= 0 + \frac{1}{S_t}(\alpha S dt + \sigma S dW) - \frac{1}{2S_t^2}\sigma^2 S_t^2 dt$

$f(t,x) = \underline{\underline{\ln x}}$

$\partial_t f = 0 \quad \partial_x f = \frac{1}{x} \quad \partial_x^2 f = -\frac{1}{x^2}.$

$d[S,S] = \sigma^2 S_t^2 \, dt$

$$= \alpha\, dt + \sigma\, dW - \frac{\sigma^2}{2} dt$$

$$dY = \left(\alpha - \frac{\sigma^2}{2}\right) dt + \sigma\, dW \qquad (\alpha, \sigma \text{ const})$$

$$Y_t - Y_0 = \left(\alpha - \frac{\sigma^2}{2}\right) t + \sigma W_t$$

$$\ln\left(\frac{S_t}{S_0}\right) = \left(\alpha - \frac{\sigma^2}{2}\right) t + \sigma W_t \Rightarrow S_t = S_0 \exp\left(\left(\alpha - \frac{\sigma^2}{2}\right) t + \sigma W_t\right)$$

(Note $\ln x$ is not diff at $x = 0$, but Ito works because $S_t > 0 \ \forall t \geq 0$).

$$dS_t = \alpha S_t \, dt + \sigma S_t \, dW_t$$

Market Assumptions.

- 1 stock. Price S_t, modelled by $\mathrm{GBM}(\alpha, \sigma)$.
- Money market: Continuously compounded interest rate r.
  - ▷ C_t = cash at time $t = C_0 e^{rt}$. (Or $\partial_t C_t = r C_t$.)
  - ▷ Borrowing and lending rate are both r.
- Frictionless (no transaction costs)
- Liquid (fractional quantities can be traded)

Consider a security that pays $V_T = g(S_T)$ at maturity time T.

Theorem 8.3. *If the security can be replicated, and $f = f(t, x)$ is a function such that the wealth of the replicating portfolio is given by $X_t = f(t, S_t)$, then:*

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$$\partial_t f + rx\partial_x f + \frac{\sigma^2 x^2}{2}\partial_x^2 f - rf = 0 \qquad x > 0,\ t < T\,, \tag{8.1}$$

(B-SM PDE)

$$f(t, 0) = g(0)e^{-r(T-t)} \qquad t \leqslant T\,, \tag{8.2}$$

(Boundary condition)

$$f(T, x) = g(x) \qquad x \geqslant 0\,. \tag{8.3}$$

(Terminal condition)

Theorem 8.4. *Conversely, if f satisfies (8.1)–(8.3) then the security can be replicated, and $X_t = f(t, S_t)$ is the wealth of the replicating portfolio at any time $t \leqslant T$.*

Remark 8.5. Wealth of replicating portfolio equals the arbitrage free price.

Remark 8.6. $g(x) = (x - K)^+$ is a European call with strike K and maturity T.

Remark 8.7. $g(x) = (K - x)^+$ is a European put with strike K and maturity T.

Proposition 8.8. *A standard change of variables gives an explicit solution to* (8.1)–(8.3)*:*

$$f(t,x) = \int_{-\infty}^{\infty} e^{-r\tau} g\Big(x \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)\tau + \sigma\sqrt{\tau}\, y\Big)\Big) \frac{e^{-y^2/2}dy}{\sqrt{2\pi}}\,, \qquad \tau = T - t\,. \tag{8.4}$$

Corollary 8.9. *For European calls, $g(x) = (x - K)^+$, and*

$$f(t,x) = c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)) \tag{8.5}$$

where

$$d_\pm(\tau,x) \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big)\,, \tag{8.6}$$

$\tau = T - t$

and

$$N(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\,dy\,, \tag{8.7}$$

$= P(N(0,1) < x)$

is the CDF of a standard normal variable.

Remark 8.10. Equation (8.1) is called a *partial differential equation.* In order to have a unique solution it needs:

PDE

(1) A terminal condition (this is equation (8.3)),
(2) A boundary condition at $x = 0$ (this is equation (8.2)),
(3) A boundary condition at infinity (not discussed yet).
   - ▷ For put options, $g(x) = (K - x)^+$, the boundary condition at infinity is
$$\lim_{x\to\infty} f(t,x) = 0\,.$$
   - ▷ For call options, $g(x) = (x - K)^+$, the boundary condition at infinity is
$$\lim_{x\to\infty} \left[f(t,x) - (x - Ke^{-r(T-t)})\right] = 0 \qquad \text{or} \qquad \boxed{f(t,x) \approx (x - Ke^{-r(T-t)}) \quad \text{as } x \to \infty\,.}$$

Expect S_t is $\gg K$, $S_T \gg K$ & payoff is $\underbrace{(S_T - K)}_{\text{forward contract.}}$

Definition 8.11. If X_t is the wealth of a self-financing portfolio then

$$\boxed{dX_t = \Delta_t\, dS_t + r(X_t - \Delta_t S_t)\, dt}$$

for some adapted process Δ_t (called the trading strategy).

Disc time : Self fin $\longrightarrow$ no ext cash flows

$$X_{t+\delta t} = \Delta_t S_{t+\delta t} + (X_t - \Delta_t S_t)(1 + r\,\delta t)$$

Position at time t : Δ_t shares of stock

Rest cash.

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$$\Rightarrow \quad X_{t+\delta t} - X_t = \Delta_t \left(S_{t+\delta t} - S_t \right) + \left(X_t - \Delta_t S_t \right) r \, \delta t$$

Send $\delta t \to 0$:
$$\underbrace{dX_t}_{} = \underbrace{\Delta_t}_{} \, \underbrace{dS_t}_{} + \left(X_t - \Delta_t S_t \right) r \, dt$$

Proof of Theorem 8.3. Assume X_t = wealth of Rep port

$= f(t, S_t)$

NTS f satisfies the BSM PDE

$X_t = f(t, S_t)$. Know $dX_t = \Delta_t \, dS_t + (X_t - \Delta_t S_t) r \, dt$

$$dS_t = \alpha S \, dt + \sigma S_t \, dW_t$$

$$\Rightarrow dX = \Delta_t\left(\alpha S\, dt + \sigma S\, dW\right) + r\left(X_t - \Delta_t S_t\right) dt$$

$$\boxed{\circledast \quad dX_t = \left(rX_t + (\alpha - r)\Delta_t S_t\right) dt + \sigma \Delta_t S_t\, dW_t}$$

Also $X_t = f(t, S_t)$

$\Rightarrow$ Itô: $dX_t = \partial_t f\, dt + \partial_x f\, dS + \frac{1}{2}\partial_x^2 f\, d[S,S]$

$$= \partial_t f \, dt + \partial_x f \left(\alpha S \, dt + \sigma S \, dW \right) + \frac{1}{2} \partial_x^2 f \, \sigma^2 S^2 dt$$

$$\boxed{(**) \quad dX = \left(\partial_t f + \alpha S \partial_x f + \frac{1}{2} \sigma^2 S^2 \partial_x^2 f \right) dt + \sigma S \partial_x f \, dW}$$

Uniqueness of S. Mg decomp $\Longrightarrow$ the dt terms & dW terms in (*) & (**) have to be equal.

$$\Rightarrow \sigma S \partial_x f = \sigma S \Delta_t \Rightarrow \partial_x f = \Delta_t$$

$$\left(\text{Note } \partial_x f = \partial_x f(t, S_t) \right)$$

Delta Hedging Rule $\longrightarrow$ $\boxed{\Rightarrow \partial_x f(t, S_t) = \Delta_t \quad (***)}$

Equate dt terms in $(*)$ & $(**)$:

Get $\partial_t f + \alpha S \partial_x f + \frac{1}{2}\sigma^2 S^2 \partial_x^2 f = r X_t + (\alpha - r)\Delta_t S_t$

Know $X_t = f(t, S_t)$ & $\delta_t = \partial_x f = \partial_x f(t, S_t)$

$$\Rightarrow \partial_t f + \alpha S \partial_x f + \frac{1}{2} \sigma^2 S^2 \partial_x^2 f = r f + (\alpha - r) S \partial_x f$$

$$\boxed{\Rightarrow \partial_t f + r S \partial_x f + \frac{\sigma^2}{2} S^2 \partial_x^2 f = r f}$$

Write x instead of S & get BSM PDE!!

Proof of Theorem 8.4.

Say f solves BS PDE

NTS $f(t, S_t) =$ wealth of R. port

Let $X_t =$ wealth of a self fin port with $X_0 = f(0, S_0)$

& $\Delta_t = \partial_x f(t, S_t)$

Set $Y_t = e^{-rt} X_t$

Know $dX_t = (r X_t + (\alpha - r) S_t)\, dt + \sigma \Delta_t S_t\, dW$

$$\boxed{\text{Choose } \Delta_t = \partial_x f(t, S_t) \quad (\text{Delta Hedging})}$$

Set $Y = e^{-rt} X_t$

$\Rightarrow$ By Itô, $dY = -r e^{-rt} X_t\, dt + e^{-rt} dX + 0$

$$\Rightarrow dY = -rY + e^{-rt}\left(rX_t + \Delta_t(\alpha - r)S_t\right)dt + e^{-rt}\sigma\,\Delta_t S_t\,dW$$

$$\boxed{\Delta_t = \partial_x f} \rightarrow \boxed{dY_t = e^{-rt}\Delta_t(\alpha-r)S_t\,dt + e^{-rt}\sigma\,\Delta_t S_t\,dW_t}$$

(2) Compute $d\left(e^{-rt}f(t,S_t)\right)$

$$= \left(e^{-rt}\partial_t f - r e^{-rt} f\right) dt + e^{-rt}\partial_x f \, dS + \frac{1}{2} e^{-rt}\partial_x^2 f \, d[S,S]$$

$$= \left(e^{-rt}\partial_t f - r e^{-rt} f\right) dt + e^{-rt}\partial_x f \left(\alpha S dt + \sigma S dW\right) + \frac{1}{2} e^{-rt}\partial_x^2 f \, \sigma^2 S^2 dt$$

$$= \left(e^{-rt}\partial_t f - r e^{-rt} f + e^{-rt}\partial_x f \cdot \underline{\alpha} S + \frac{1}{2}\sigma^2 S^2 e^{-rt}\partial_x^2 f\right) dt$$

$$+ e^{-rt} \partial_x f \, S_t \, \sigma \, dW$$

$$= e^{-rt} \underbrace{\partial_x f} \cdot (\alpha - r) S_t \, dt + \underbrace{e^{-rt} \partial_x f \, \sigma \, S_t} \, dW$$

$$= dY \Rightarrow d\left(e^{-rt} f(t, S_t)\right) = d\left(e^{-rt} X_t\right)$$

Choose $X_0 = f(0, S_0)$

$\Rightarrow$ ~~$\forall$~~ for all $t < T$,

$$e^{-rt} f(t, S_t) = e^{-rt} X_t$$

$$\boxed{\Rightarrow f(t, S_t) = X_t}$$