- No ~~arti~~ arbitrage.

$X_0 = 0, \quad X_N \geq 0 \Rightarrow \mathbb{P}(X_N = 0) = 1$

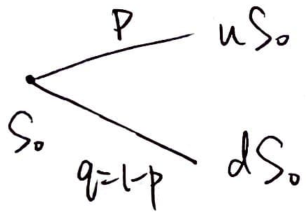
- Arbitrage free price (AFP)

security V_N

Starts X_0, $X_N = V_N$

$\Rightarrow$ AFP $= X_0$.

- Binomial model

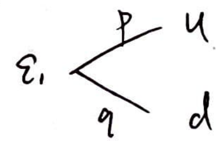


no arbitrage $\Leftrightarrow d < 1+r < u$.

e.g. $1+r = d$.

$t=0$ long 1 Stock $\qquad$ $t=1$

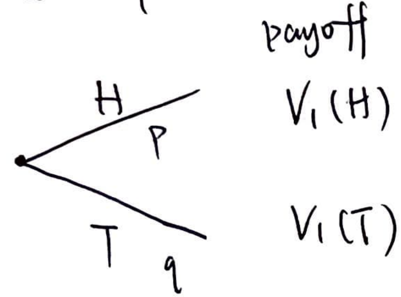
short S_0 cash $\qquad$ $S_1 = \varepsilon_1 S_0$



$X_0 = 1 \cdot S_0 - S_0 = 0 \qquad X_1 = S_1 - (1+r)S_0 = (\varepsilon_1 - \cancel{1+r}\, d) S_0 \geq 0$

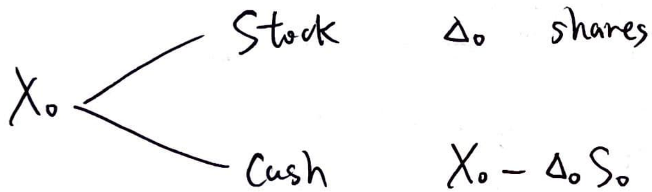
$\mathbb{P}(X_1 > 0) = \mathbb{P}(\varepsilon_1 = u) = p > 0.$

- 1 − period



$$\tilde{p} = \frac{1+r-d}{u-d}, \quad \tilde{q} = \frac{u-(1+r)}{u-d}$$

$$V_0 = \frac{1}{1+r}\left(\tilde{p}\, V_1(H) + \tilde{q}\, V_1(T)\right) = \frac{1}{1+r}\tilde{\mathbb{E}}\, V_1$$

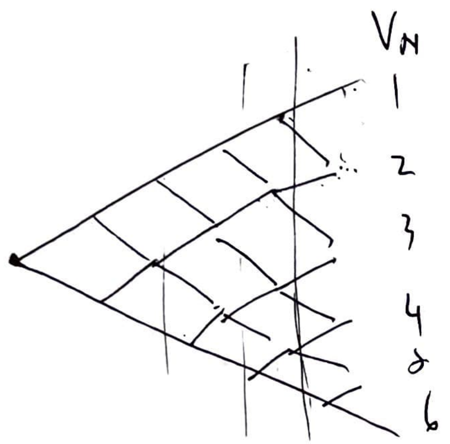


$$\Delta_0 = \frac{V_1(H) - V_1(T)}{(u-d)S_0}.$$

$$X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) = V_1$$

$$\begin{cases} V_1(H) = \Delta_0 u S_0 + (1+r)(X_0 - \Delta_0 S_0) \\ V_1(T) = \Delta_0 d S_0 + (1+r)(X_0 - \Delta_0 S_0) \end{cases}$$

- Muti - Period



$\tilde{p} = \frac{1+r-d}{u-d}$, $\tilde{q} = \frac{u-(1+r)}{u-d}$

$V_n = \frac{1}{(1+r)^{N-n}} \tilde{\mathbb{E}}_n V_N.$

- Discrete probability space. (finite PS)

$\Omega = \{ w = (w_1, \cdots, w_N) \mid w_i = H \text{ or } T \}.$

sample point $w = (w_1, \cdots, w_N) \in \Omega.$

- Probability mass function (PMF)

$p: \Omega \to [0,1]$ s.t. $\sum_{w \in \Omega} p(w) = 1.$

e.g. $p(w) = p^{H(w)} q^{T(w)}$, $H(w) = \#$ of heads in $(w_1, \cdots, w_N)$.

- Event. $A \subseteq \Omega$. $\mathbb{P}(A) = \sum_{w \in A} p(w)$. $\mathbb{P}(\{w_i = H\}) = p$

- Random variable

$X : \Omega \to \mathbb{R}$

$\text{Range}(X) = \{ X(\omega) \mid \omega \in \Omega \}$.

$\sigma(X) = \{ \{X \in A\} \mid A \subseteq \mathbb{R} \} = \{ \{X \in A\} \mid A \subseteq \text{Range}(X) \}$.

$\{X \in A\} = \{ \omega \in \Omega \mid X(\omega) \in A \}$

$\{ \omega \in \Omega \mid X(\omega) \notin \text{Range}(X) \} = \phi$.

ϕ, Ω always in $\sigma(X)$.

$(A = \phi)$ $(A = \mathbb{R}$ or $A = \text{Range}(X))$

$X(HH) = X(HT) = 1, \quad X(TH) = X(TT) = 0.$

$\sigma(X) = \{ \phi, \Omega, \{X=1\}, \{X=0\} \}$

$= \{ \phi, \Omega, \{HH, HT\}, \{TH, TT\} \}$

~~$\{\phi,$~~

$\{ \phi, \{0\}, \{1\}, \{0,1\} \}$ X

$\sigma(X) \subseteq \mathcal{P}(\Omega) = \{$ all subsets of $\Omega \}$.

- Expectation

$\{X = x\} = \{ \omega \in \Omega \mid X(\omega) = x \}$.

$$\mathbb{E}X = \sum_{\omega \in \Omega} X(\omega) p(\omega) = \sum_{x \in \text{Range}(X)} x \, \mathbb{P}(X = x)$$

2^N

- Independence of Events.

$A_1, \cdots, A_n$ events,

$A_1, \cdots, A_n$ are independent, if $\forall\ 1 \le i_1 < i_2 < \cdots < i_k \le n$,

$\mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k}) = \mathbb{P}(A_{i_1}) \cdot \cdots \cdot \mathbb{P}(A_{i_k})$.

- $\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\mathbb{P}(A_j)$, $\forall i<j$ $\times$ $\cancel{\Rightarrow}$

- $\mathbb{P}(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_1) \cdot \cdots \cdot \mathbb{P}(A_n)$ $\times$. $\quad \mathbb{P}(A_1^c \cap A_2 \cap \cdots \cap A_n) = \mathbb{P}(A_1^c)\mathbb{P}(A_2) \cdots \mathbb{P}(A_n)$

Ex 1. $\Omega = \{1, 2, 3, 4\}$, $\mathbb{P}(\{\omega\}) = \frac{1}{4}$, $\omega \in \Omega$.

$A_1 = \{1, 2\}$, $A_2 = \{1, 3\}$, $A_3 = \{1, 4\}$.

$\mathbb{P}(A_i) = \frac{1}{2}$, $\forall i$

$\mathbb{P}(A_i \cap A_j) = \mathbb{P}(\{1\}) = \frac{1}{4}$ $\quad \forall i \neq j$

$= \mathbb{P}(A_i)\mathbb{P}(A_j)$

$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(\{1\}) = \frac{1}{4} \neq \frac{1}{8} = \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3)$

Ex 2. $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $\mathbb{P}(\{\omega\}) = \frac{1}{8}$.

$A = B = \{1, 2, 3, 4\}$, $C = \{1, 5, 6, 7\}$. $\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C) = \frac{1}{2}$.

$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(\{1\}) = \frac{1}{8} = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$

$\mathbb{P}(A \cap B) = \mathbb{P}(A) = \frac{1}{2} \neq \frac{1}{4} = \mathbb{P}(A)\mathbb{P}(B)$

~~$\mathbb{P}(A$~~

- $X_1, \cdots, X_n$ R.V. are independent if

$$\mathbb{P}(X_1 = x_1, \cdots, X_n = x_n) = \mathbb{P}(X_1 = x_1) \cdot \cdots \cdot \mathbb{P}(X_n = x_n), \quad \forall x_i \in \text{Range}(X_i).$$

Fix $x_2, x_3, \cdots, x_n$,

$$\sum_{x_1 \in \text{Range}(X_1)} \mathbb{P}(X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n) = \underbrace{\sum_{x_1 \in \text{Range}(X_1)} \mathbb{P}(X_1 = x_1)\left(\mathbb{P}(X_2 = x_2) \cdots \mathbb{P}(X_n = x_n)\right)}_{= \mathbb{P}(X_2 = x_2) \cdots \mathbb{P}(X_n = x_n)}$$

$$= \mathbb{P}\Big(\bigcup_{x_1 \in \text{Range}(X_1)} \{X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n\}\Big)$$

$$= \mathbb{P}(X_2 = x_2, \cdots, X_n = x_n) = \mathbb{P}(X_2 = x_2) \cdots \mathbb{P}(X_n = x_n)$$

- Filtration

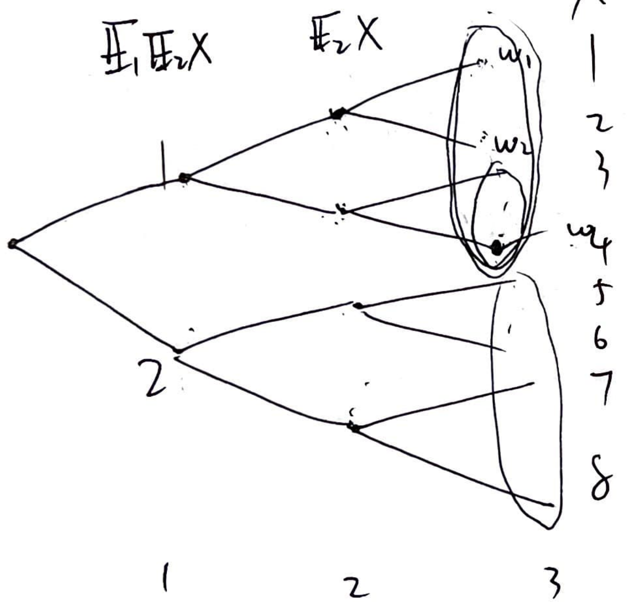
$$\{\phi, \Omega\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_N = \mathcal{P}(\Omega).$$

X_n is $\mathcal{F}_n$-measurable if X_n only depends on $\omega_1, \cdots, \omega_n$.

$X_n : \Omega \to \mathbb{R}$.

$$\frac{\mathbb{P}(\{\omega_1\})}{\mathbb{P}(\{\omega_1, \omega_2\})} \qquad \frac{\mathbb{P}(\{\omega_2\})}{\mathbb{P}(\{\omega_1, \omega_2\})}$$

$\omega \in \Omega$, $\omega = (\omega_1, \cdots, \omega_N)$



e.g. X is $\mathcal{F}_1$-measurable

$\Pi_1(\omega)$ $\Pi_2(\omega)$

$\mathbb{E}_1 X$

X is independent of $\mathcal{F}_n$ if

$$\mathbb{P}(\underline{\{X \in A\}} \cap B) = \mathbb{P}(X \in A)\,\mathbb{P}(B), \quad \underline{A} \subseteq \underline{\mathbb{R}}, \quad B \in \mathcal{F}_n.$$

- Conditional expectation

$$\mathbb{E}_n X(\omega) = \mathbb{E}(X|\mathcal{F}_n)(\omega) = \sum_{x \in \mathrm{Range}(X)} x\, \mathbb{P}(X=x \mid \Pi_n(\omega)), \quad = \frac{\mathbb{P}(\{X=x\} \cap \Pi_n(\omega))}{\mathbb{P}(\Pi_n(\omega))}$$

it is a R.V.

where $\Pi_n(\omega) = \{ \omega' \in \Omega \mid \omega_1' = \omega_1, \omega_2' = \omega_2, \cdots, \omega_n' = \omega_n \}$.

- $\mathbb{E}_n X$ is $\mathcal{F}_n$ - measurable

$\mathbb{E}_n(\alpha X + \beta Y) = \alpha \mathbb{E}_n X + \beta \mathbb{E}_n Y$

$m \leq n$, $\mathbb{E}_m(X) = \mathbb{E}_m(\mathbb{E}_n X) = \mathbb{E}_n(\mathbb{E}_m X)$

X is $\mathcal{F}_n$-measurable, $\mathbb{E}_n(XY) = X\mathbb{E}_n Y$ $\quad (Y=1 \Rightarrow \mathbb{E}_n X = X)$

$\mathbb{E}\,\mathbb{E}_n X = \mathbb{E} X$

- $X_0, X_1, \cdots, X_N$ is a stochastic process.

Adapted : $\forall n$, X_n is $\mathcal{F}_n$ - measurable.

Martingale : adapted + $\mathbb{E}_n X_{n+1} = X_n$, $\forall n \geq 0$.

Ex. symmetric random walk, $X_n = \xi_1 + \cdots + \xi_n$, ξ_i i.i.d. $\mathbb{P}(\xi_i = \pm 1) = \frac{1}{2}$.

- Brownian motion W_t $\quad \Omega = C([0,\infty))$

W_t is a continuous process (fix $\omega \in \Omega$, $W_t(\omega)$ as a function of t is continuous on $[0,\infty)$).

s.t. (1) $W_0 = 0$

(2) $\forall\, 0 \le s < t$, $W_t - W_s \sim N(0, t-s)$ $\quad$ ($Var(W_t - W_s) = t-s \ne (t-s)^2$)

(3) $W_t - W_s$ is independent of $\mathcal{F}_s$ $\quad$ ($\mathbb{P}(\{W_t - W_s \in A\} \cap B) = \mathbb{P}(W_t - W_s \in A)\,\mathbb{P}(B)$).

$\forall\, 0 \le s < t$

- Sample space $\Omega = C([0,\infty))$. not finite, not even countable.

$\Omega \ne \{\omega_1, \omega_2, \cdots, \omega_n, \cdots\}$

$\mathcal{G} \subseteq \mathcal{P}(\Omega)$ $\quad$ (1° $\phi \in \mathcal{G}$ $\quad 2^\circ$ $A \in \mathcal{G}$ implies $A^c \in \mathcal{G}$ $\quad 3^\circ$ $A_1, A_2, \cdots, A_n, \cdots \in \mathcal{G}$ implies $\bigcup_{n=1}^{\infty} A_n \in \mathcal{G}$)

Define $\mathbb{P} : \mathcal{G} \to [0,1]$ $\quad$ (1° $\mathbb{P}(\phi) = 0$, 2° if $A \cap B = \phi$, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

3° if $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$, $\mathbb{P}(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mathbb{P}(A_n)$)

X is a R.V. $\quad X : \Omega \to \mathbb{R}$

$\mathbb{E}X = \int_\Omega X(\omega)\, d\mathbb{P}(\omega)$ $\quad$ in general, no formula. $\quad$ (if X has a pdf p $\quad \mathbb{E}X = \int_{\mathbb{R}} x\, p(x)\, dx$)

$X = \mathbb{1}_A \qquad \mathbb{1}_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A. \end{cases}$

$\mathbb{E}\mathbb{1}_A = \int_\Omega \mathbb{1}_A(\omega)\, d\mathbb{P}(\omega) = \mathbb{P}(A)$. $\qquad \mathbb{E}\left(\sum_{i=1}^{n} a_i \mathbb{1}_{A_i}\right) = \sum_{i=1}^{n} a_i\, \mathbb{P}(A_i)$

- $\mathcal{F}_t = \{$ events that can be described by $(B_s)_{0\le s\le t}\}$.

  $\{W_1 \in (1,2)\} \in \mathcal{F}_2$.

  $\{W_3 > 0\} \notin \mathcal{F}_2$

- martingale. X_t

  1° X is adapted

  2° $\mathbb{E}_s X_t = X_s$, $\forall\, 0 \le s < t$ $\quad$ $(\mathbb{E}_t X_{t+1} = X_t,\ \forall t \ge 0$ ✗$)$

  $\mathbb{E}_1 X_{3/2} = ?$

- W is a BM

  $\forall\ 0 < t_1 < t_2 < \cdots < t_n$

  $(W_{t_1}, W_{t_2}, \cdots, W_{t_n})$ is joint normal

  $$\begin{pmatrix} W_{t_1} \\ W_{t_2} \\ \vdots \\ W_{t_n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1 \end{pmatrix} \underbrace{\begin{pmatrix} W_{t_1} \\ W_{t_2} - W_{t_1} \\ \vdots \\ W_{t_n} - W_{t_{n-1}} \end{pmatrix}}_{\text{joint normal}}$$

  $\mathbb{E}(W_{t_i} W_{t_j}) = t_i \wedge t_j = \min\{t_i, t_j\}$.

Ex. Not every two normal r.v.'s form a joint normal distribution.

$X \sim N(0,1)$, $\varepsilon \begin{cases} 1 & 1/2 \\ -1 & 1/2 \end{cases}$, X, ε independent, $Y = \varepsilon X$

$$\mathbb{P}(Y \le y) = \mathbb{P}(\varepsilon X \le y) = \mathbb{P}(\varepsilon X \le y, \varepsilon = 1) + \mathbb{P}(\varepsilon X \le y, \varepsilon = -1)$$

$= \mathbb{P}(X \leq y, \varepsilon = 1) + \mathbb{P}(-X \leq y, \varepsilon = -1)$

$= \mathbb{P}(X \leq y)\,\underline{\mathbb{P}(\varepsilon = 1)} + \mathbb{P}(-X \leq y)\,\underline{\mathbb{P}(\varepsilon = -1)}$

$= \frac{1}{2}\left(\mathbb{P}(X \leq y) + \mathbb{P}(-X \leq y)\right)$ $\quad (X \sim N(0,1), \quad -X \sim N(0,1))$

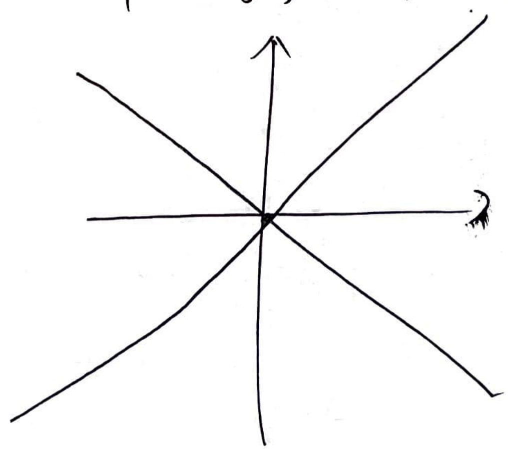
$= \frac{1}{2}\left(\Phi(y) + \Phi(y)\right) = \Phi(y).$

$\Rightarrow Y = \varepsilon X \sim N(0,1)$

(X, Y) is not joint normal.

$= (X, \varepsilon X)$

$\text{Range}(X, \varepsilon X) = \{(x,y) \in \mathbb{R}^2 \mid x = y\} \cup \{(x,y) \in \mathbb{R}^2 \mid x = -y\} \neq \mathbb{R}^2.$



$(X, \varepsilon X)$ not joint normal.

$\text{Cov}(X, Y) = \mathbb{E}XY = \mathbb{E}(\varepsilon X^2)$

$= \underline{\mathbb{E}\varepsilon} \cdot \mathbb{E}X^2 = 0$

X, Y uncorrelated.

X, Y not independent.