7. Review Problems

$W \to$ B.M.

Problem 7.1. If $0 \leqslant r \leqslant s \leqslant t$, find $\boldsymbol{E}(W_s W_t)$ and $\boldsymbol{E}(W_r W_s W_t)$.

Guess $E W_r^3 = 0$

$$E(W_s W_t) = s \wedge t \qquad (\min\{s, t\})$$

$$= s$$

$$\left(\text{Sol 1:} \quad E(W_s W_t) \overset{\text{tower}}{=} E\, E_s(W_s W_t) = E\big(W_s \underbrace{E_s W_t}_{Mg}\big) \right.$$

$$\left. = E(W_s W_s) = s \qquad (W_s \sim N(0, s)) \right)$$

Sol 2: $\mathbb{E} W_s W_t = \mathbb{E} W_s (W_s + W_t - W_s)$

$$= \underbrace{\mathbb{E} W_s^2}_{= s} + \underbrace{\mathbb{E} W_s (W_t - W_s)}_{0}$$

($\because$ $W_t - W_s$ is ind of W_s

$W_t - W_s \sim N(0, t-s)$)

Compute $\mathbb{E}(W_r W_s W_t)$ $\quad r \le s \le t$

$$= E\left(W_r \, E_s \left(W_s W_t \right) \right)$$

$$= E\left(W_r W_s \, E_s W_t \right) = E\left(W_r W_s^2 \right)$$

$$= E \, E_r\left(W_r W_s^2 \right) = E\left(W_r \, E_r W_s^2 \right)$$

$$= E\left(W_r \, E_r\left(W_s^2 - s + s \right) \right)$$

$$= E\left(W_r \left(W_r^2 - r + s \right) \right) \quad \left(\because W_s^2 - s \text{ is a mg} \right)$$

$$= E W_r^3 + E \underbrace{W_r (s-r)}_{} = 0$$

Problem 7.2. Define the processes X, Y, Z by

$$X_t = \int_0^{W_t} e^{-s^2}\,ds, \quad Y_t = \exp\left(\int_0^t W_s\,ds\right), \quad Z_t = tX_t^2$$

Decompose each of these processes as the sum of a martingale and a process of finite first variation. What is the quadratic variation of each of these processes?

(1) Write $X = X_0 + \underbrace{B}_{BV} + \underbrace{M}_{Mg}$

Usual strategy: $X_t = f(t, W_t)$ & apply Itô

$$dX = (\quad)\,dt + (\quad)\,dW$$

BV part Mg part.

Let $f(t,x) = \int_0^x e^{-s^2}\, ds \Rightarrow X_t = f(t, W_t)$

$\partial_t f = 0$

$\partial_x f = e^{-x^2}$. (FTC)

$\partial_x^2 f = -2x\, e^{-x^2}$ (Chain rule)

d

$$dX_t = d\,f(t, X_t) = \partial_t f\, dt + \partial_x f\, dW + \frac{1}{2}\partial_x^2 f\, \underbrace{d[W, W]}_{dt}$$

$$= 0\, dt + e^{-W_t^2} dW - \frac{1}{2} 2 W_t e^{-W_t^2} dt$$

$$= \left(-W_t e^{-W_t^2} dt\right) + \left(e^{-W_t^2} dW_t\right)$$

$$\Rightarrow X_t = X_0 + \underbrace{\int_0^t -W_s e^{-W_s^2} \, ds}_{} + \int_0^t e^{-W_r^2} \, dW_r$$

$$X_0 = 0 \qquad B_t = -\int_0^t W_s e^{-W_s^2} \, ds$$

$$M_t = \int_0^t e^{-W_r^2} \, dW_r$$

$$Y_t = \text{ext}\left(\int_0^{t} W_s\, ds\right) \longleftarrow \text{already a diff fn of } t$$

(finite 1st var)

Note $g(t) = \int_0^t W_s\, ds \longrightarrow$ diff fn of t

$$Y_t = Y_0 + \underbrace{(Y_t - Y_0)}_{\text{finite 1st var (because } Y \text{ is diff)}} + 0$$

$$Z_t = f(t, X_t) \quad , \quad f(t,x) = t\, x^2 .$$

& Just Ito to decompose Z

Problem 7.3. Define the processes X, Y by

$$X_t \stackrel{\text{def}}{=} \int_0^t W_s \, ds \,, \quad Y_t \stackrel{\text{def}}{=} \int_0^t W_s \, dW_s \,.$$

Given $0 \leqslant s < t$, compute $\boldsymbol{E} X_t$, $\boldsymbol{E} Y_t$, $\boxed{\boldsymbol{E}_s X_t}$, $\boldsymbol{E}_s Y_t$.

Notation

$$\left(E_s X_t = \underbrace{E(X_t \mid \mathcal{F}_s)} \right)$$

(1) $E \displaystyle\underbrace{\int_0^t W_s \, ds}_{\text{Riemann int}} = \int_0^t E W_s \, ds = 0$

$E \int dt = \int E \, dt$

(2) $$E_s \int_0^t W_r \, dr = \int_0^t E_s W_r \, dr$$

(Riemann Int)

$$= \int_0^s E_s W_r \, dr + \int_s^t E_s W_r \, dr$$

$$= \int_0^s W_r \, dr + \int_s^t W_s \, dr$$

$$= \int_0^s W_r \, dr + W_s (t-s)$$

② $Y_t = \int_0^t W_s \, dW_s$. Find EY_t & $E_s Y_t$

ⓐ $$E \underbrace{\int_0^t W_s \, dW_s}_{M_g} = \int_0^0 W_s \, dW_s = 0$$

(b) $E_s Y_t = E(Y_t \mid \mathcal{F}_s) = E_s \int_0^t W_r \, dW_r$

$$\overset{Mg}{=} \int_0^s W_s \, dW_s$$

Ito integrals are Mg's

QV of $\int_0^t \sigma_s \, dW_s$ is $\int_0^t \sigma_s^2 \, ds$

& $E\left(\int_0^t \sigma_s \, dW_s\right)^2 \overset{\text{Ito isom}}{=} E \int_0^t \sigma_s^2 \, ds$

Problem 7.4. Let $M_t = \int_0^t W_s \, dW_s$. Find a function f such that

$$\mathcal{E}(t) \stackrel{\text{def}}{=} \exp\Big(M_t - \int_0^t f(s, W_s)\, ds\Big)$$

is a martingale.

Let $g(t, x) = \exp\left(x - \int_0^t f(s, W_s)\, ds\right)$

$\mathcal{E}(t) = g(t, M_t)$

① $\partial_t g = \exp(\quad) \cdot (-f(t, W_t))$

② $\partial_x g = \exp(\quad)\, 1$

③ $\partial_x^2 g = \exp(\quad) \cdot 1$

④ $d[M,M]_t = W_t^2 \, dt$

$$\Rightarrow d\mathcal{E}(t) = \partial_t f \, dt + \partial_x f \, dM + \frac{1}{2}\partial_x^2 f \, d[M,M]$$

$$= \mathcal{E}(t)\left[-f(t,W_t)\right] dt + \mathcal{E}(t) W_t \, dW_t + \frac{1}{2}\mathcal{E}(t) W_t^2 \, dt$$

$$= \mathcal{E}(t) \underbrace{\left[-f(t, W_t) + \frac{1}{2} W_t^2 \right]}_{} dt + \mathcal{E}(t) W_t \, dW_t$$

Choose f so that the dt term vanishes

$$\boxed{\Rightarrow f(t,x) = \frac{x^2}{2}}$$

i.e. $\mathcal{E}(t) = \exp\left(\int_0^t W_s \, dW_s - \frac{1}{2} \int_0^t W_s^2 \, ds \right)$

is a mg

Problem 7.5. Suppose $\sigma = \sigma_t$ is a deterministic (i.e. non-random) process, and M is a martingale such that $d[M, M]_t = \sigma_t^2 \, dt$. (Say $M_0 = 0$)

$$\cancel{X_t = \int_0^t \sigma_u \, dW_u}\,.$$

(1) Given $\lambda, s, t \in \mathbb{R}$ with $0 \leqslant s < t$ compute $\boldsymbol{E} e^{\lambda M_t}$ and $\boldsymbol{E}_s e^{\lambda M_t - M_s}$
(2) If $r \leqslant s$ compute $\boxed{\boldsymbol{E} \exp(\lambda M_r + \mu(M_t - M_s))}$.
(3) What is the joint distribution of $(M_r, M_t - M_s)$?
(4) *(Lévy's criterion)* If $\boxed{d[M, M]_t = dt}$, then show that M is a standard Brownian motion.

Compute $E\, e^{\lambda M_t} = (\text{Mgf of } M_t)$

$$f(t,x) = e^{\lambda x} . \qquad \partial_t f = 0, \quad \partial_x f = \lambda e^{\lambda x}, \quad \partial_x^2 f = \lambda^2 e^{\lambda x}$$

$$d[M,M] = \sigma_t^2 \, dt$$

Let $\underline{\underline{\varphi(t) = E e^{\lambda M_t}}}$

$$d\left(e^{\lambda M_t}\right) \overset{Ito}{=\!=} \partial_t f \, dt + \partial_x f \, dM + \frac{1}{2} \partial_x^2 f \, d[M,M]$$

$$= 0 + \lambda e^{\lambda M_t} dM + \frac{1}{2} \lambda^2 e^{\lambda M_t} \sigma_t^2 \, dt$$

$$\Rightarrow \quad e^{\lambda M_t} - \underbrace{e^{\lambda M_0}}_{1} = \lambda \int_0^t e^{\lambda M_s}\, dM_s + \frac{1}{2}\lambda^2 \int_0^t e^{\lambda M_s} \sigma_s^2\, ds$$

$$\Rightarrow \quad E e^{\lambda M_t} - 1 = \lambda E \underbrace{\int_0^t e^{\lambda M_s}\, dM_s}_{\substack{M_g \\ E(\,)=0}} + \frac{1}{2}\lambda^2 E \int_0^t e^{\lambda M_s} \sigma_s^2\, ds$$

Riem Int

nat random.

$$\varphi(t) = E\, e^{\lambda M_t}$$

$$\Rightarrow \varphi(t) = 1 + 0 + \frac{1}{2}\lambda^2 \int_0^t \varphi(s)\, \sigma_s^2 \, ds$$

$$\Rightarrow \varphi'(t) = \frac{\lambda^2}{2} \varphi(t)\, \sigma_t^2$$

$$\Rightarrow \frac{\varphi'}{\varphi} = \frac{\lambda^2}{2} \sigma_t^2 \Rightarrow \frac{d}{dt}\left(\ln \varphi\right) = \frac{\lambda^2}{2} \sigma_t$$

$$\Rightarrow \quad \underbrace{\ln \varphi(t) - \ln(\varphi(0))}_{\ln\left(\frac{\varphi(t)}{\varphi(0)}\right)} = \frac{\lambda^2}{2} \int_0^t \sigma_s^2 \, ds$$

$$\Rightarrow \quad \varphi(t) = \varphi(0) \exp\left(\frac{\lambda^2}{2} \int_0^t \sigma_s^2 \, ds \right)$$

$$\Rightarrow \underbrace{E\, e^{\lambda M_t}}_{\text{Mgf of } M_t} = 1 \cdot \underbrace{\exp\left(\frac{\lambda^2}{2}\int_0^t \sigma_s^2\, ds\right)}_{\text{Mgf of } N\left(0,\ \int_0^t \sigma_s^2\, ds\right)}$$

$$\Rightarrow M_t \sim N\left(0,\ \int_0^t \sigma_s^2\, ds\right)$$

Use the same trick to compute

$$E_s\, e^{\lambda(M_t - M_s)}$$

$$E_s\, e^{\lambda M_t} - e^{\lambda M_s} = \lambda\, E_s \int_s^t e^{\lambda M_r}\, dM_r + \frac{1}{2}\lambda^2\, E_s \int_s^t e^{\lambda M_r}\, \sigma_r^2\, dr$$

Get $E_s\, e^{\lambda(M_t - M_s)} = \exp\left(\frac{\lambda^2}{2} \int_s^t \sigma_r^2\, dr\right) \leftarrow$ MGF of norml.

(smells like $M_t - M_s$ shall be ind of $\mathcal{F}_s$).

Note: $\underline{\underline{X}}$ & $\underline{\underline{Y}}$ are ind $\iff$ $\underbrace{\mathbb{E}\left(e^{\lambda X + \mu Y}\right)}_{\text{Joint MGF}} = \underbrace{\mathbb{E} e^{\lambda X} \, \mathbb{E} e^{\mu Y}}_{\text{product of MGF's}}$

(for all λ, μ).

Lets compute $$\mathbb{E}\left(e^{\lambda M_r + \mu(M_t - M_s)}\right) \qquad (r \le s \le t)$$

$$= \mathbb{E}\, \mathbb{E}_s \left(\qquad \right)$$

$$= \mathbb{E}\, \mathbb{E}_s \left(e^{\lambda M_r} e^{\mu(M_t - M_s)} \right) = \mathbb{E}\left(e^{\lambda M_r} \underbrace{\mathbb{E}_s e^{\mu(M_t - M_s)}} \right)$$

$$= E\left(e^{\lambda M_r} \cdot e^{\frac{\mu^2}{2}\int_s^t \sigma_u^2\,du} \right)$$

$$= e^{\frac{\lambda^2}{2}\int_0^r \sigma_u^2\,du + \frac{\mu^2}{2}\int_s^t \sigma_u^2\,du}$$

$$= \text{MGF of a 2D normal mean } 0 \text{ \& covariance } \begin{pmatrix} \int_0^r \sigma_u^2\,du & 0 \\ 0 & \int_s^t \sigma_u^2\,du \end{pmatrix}$$

$$\boxed{\Rightarrow M_t - M_s \text{ is ind of } M_r.}$$

If ~~Put~~ $r=1$ Get $M_t - M_s \sim N(0, t-s)$ & ind of $\mathcal{F}_s$ $\Rightarrow M$ is a BM.

(Levy's Criterion)

Problem 7.6. Define the process X, Y by

$$X = \int_0^t s\, dW_s\,, \quad Y = \int_0^t W_s\, ds\,.$$

Find a formula for $\boldsymbol{E} X_t^n$ and $\boldsymbol{E} Y_t^n$ for any $n \in \mathbb{N}$.

Claim: Both X & Y are Normal!

not random.

$$X_t = \lim \underbrace{\sum t_i (W_{t_{i+1}} - W_{t_i})}_{\text{linear combination of Normals!}} \Rightarrow X \text{ is Normal.}$$

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To find EX_t^n just find EX_t & EX_t^2

& use the formula for moments of Normal RV's.

$$EX_t = E\int_0^t s\,dW_s = 0$$

$$EX_t^2 = E\left(\int_0^t s\,dW_s\right)^2 \underset{\text{Ito Isom}}{=} E\int_0^t s^2\,ds = \frac{t^3}{3}.$$

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Same for Y:

$$Y_t = \int_0^t W_s \, ds = \lim_{\|P\| \to 0} \sum \underbrace{W_{t_i}}_{\text{Normal}} \underbrace{(t_{i+1} - t_i)}_{\text{Not random.}}$$

Linear comb of Normal

$\Rightarrow Y$ is Normal.

① $$EY_t = E\int_0^t W_s\,ds = \int_0^t EW_s\,ds = 0$$

② $$EY_t^2 = E\left(\int_0^t W_s\,ds\right)^2 = E\left(\int_0^t W_s\,ds\right)\left(\int_0^t W_r\,dr\right)$$

$$= E\left(\int_0^t W_s\,ds \int_0^t W_r\,dr\right) = E\int_{s=0}^t \int_{r=0}^t W_s W_r\,ds\,dr$$

$$= \int_{s=0}^{t} \int_{r=0}^{t} E(W_s W_r) \, ds \, dr$$

$$= \int_{s=0}^{t} \int_{r=0}^{t} (s \wedge r) \, ds \, dr \quad \text{\& integrate!}$$

Get Mean & Variance & use MGF of Normal to find all moments!