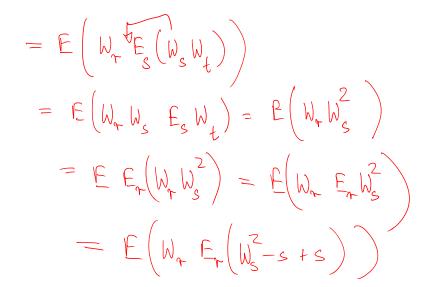
7. Review Problems
$$W \rightarrow H B.M.$$

Problem 7.1. If $0 \le r \le s \le t$, find $E(W_s W_t)$ and $E(W_r W_s W_t)$.
 $E(W_s W_t) = S \wedge t$ ($\min \{s, t\}$)
 $= S$
(Sd_{1} ; $E(W_s W_t) \stackrel{\text{favor}}{=} E \stackrel{\text{form}}{E_s} (W_s W_t) = E(W_s \stackrel{\text{Mg}}{E_s} W_t)$
 $= E(W_s W_s) = S$ ($W_s \sim N(v_s)$)

 $S_{M} 2 : P_{W_{S}} W_{t} = P_{W_{S}} (W_{s} + W_{t} - W_{s})$ $= \mathcal{E}W_{s}^{2} + \mathcal{E}W_{s}(W_{t} - W_{s})$ O ("Wy-Ws is ind of Ws $W_{t} - W_{s} \sim N(0, t-s)$ Compute E (Wy Ws Wy) rsset



 $= \mathcal{E}\left(\mathcal{W}_{r}\left(\mathcal{W}_{r}^{2}-r+s\right)\right)\left(\mathcal{W}_{s}^{2}-s \text{ is a mag}\right)$

 $= EW_{r}^{s} + EW_{r}(s-r) = O$

Problem 7.2. Define the processes X, Y, Z by

$$X_t = \int_0^{W_t} e^{-s^2} \frac{ds}{ds}, \quad Y_t = \exp\left(\int_0^{t} W_s \frac{ds}{ds}\right), \quad Z_t = tX_t^2$$

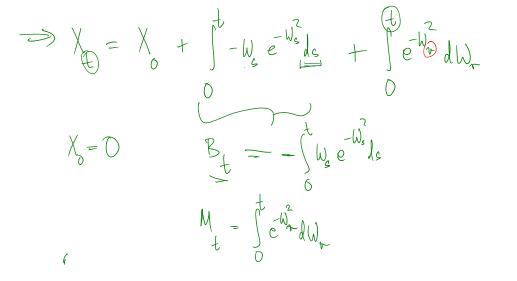
Decompose each of these processes as the sum of a martingale and a process of finite first variation. What is the quadratic variation of each of these processes?

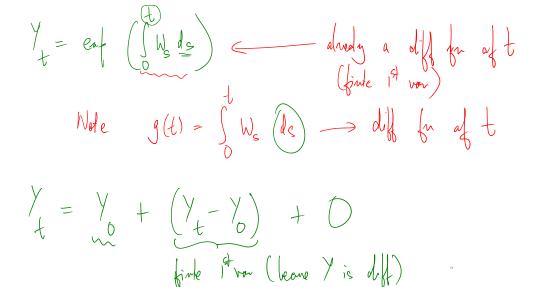
Wate
$$X = X_0 + B + M_{BV}$$

BV Mg
Usual strategy: $X_t = f(t, W_t)$ & apply Itô
 $dX = () dt + () dW$

Mg fait. BV fourt $\int e^{-s} ds$ het f(t, n) = \Rightarrow X_t = {(t, W_t) $\begin{aligned} \partial_{t}f &= D \\ \partial_{x}f &= e^{-\chi^{2}} (FTC) \\ \partial_{x}^{2}f &= -2\chi e^{-\chi^{2}} (lhain whe) \end{aligned}$

 $dX_{t} = d f(t, X_{t}) = \frac{2}{2} \int dt + \frac{2}{2} \int dW + \frac{1}{2} \frac{2}{2} \int d[W, W]$ $= 0 dt + e^{-W_t^2} dW - \frac{1}{2} 2W_t e^{-W_t^2} dt$ $= \left(-\psi_{t} \bar{e}^{\psi_{t}} dt\right) + \left(\bar{e}^{\psi_{t}} d\psi_{t}\right)$





 $\mathcal{Z}_{L} = \{(t, \chi), (t, \pi) = t \pi^{2}\}$

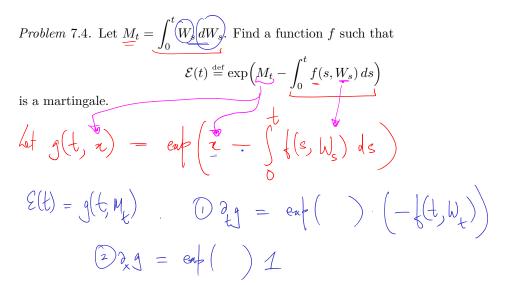
& Just Ito to decompose Z

Problem 7.3. Define the processes X, Y by $X_{t} \stackrel{\text{def}}{=} \int_{0}^{t} W_{s} \, ds \,, \quad Y_{t} \stackrel{\text{def}}{=} \int_{0}^{t} W_{s} \, dW_{s} \,. \qquad \swarrow$ $(E_{c}X) =$ Given $0 \leq s < t$, compute EX_t , EY_t , $|E_sX_t| E_sY_t$. $DE \int_{0}^{t} W_{s} ds = \int_{0}^{t} EW_{s} ds$ Eld= (E dt

 $(2) E_{s} \int_{0}^{t} W_{r} dr = \int_{0}^{t} E_{s} W_{r} dr$ (Riemen Int) $= \int E_{s}W_{r} dv + \int F_{s}W_{r} dv$ $= \int_{0}^{\infty} W_{r} dr + \int_{0}^{\infty} W_{s} dr$

 $= \int_{0}^{s} W_{r} dr + W_{s}(t-s)$ € Y₄ = J W₂ dW₅. Find EY_t & E₅Y_t $D \in \int W_{s} dW_{s} = \int W_{s} dW_{s}$ = ()

 $(b) E_{s}Y_{t} = E(Y_{t} | \xi) = E_{s}(t) W_{t} dW_{t}$ $M_{2} \int_{0}^{s} W_{s} I W_{s}$ $\int_{0}^{t} \frac{1}{\sqrt{s}} dS = E \left(\int_{0}^{t} \sqrt{s} dW_{s} \right)$ $I_{0} = E \int_{0}^{t} \sqrt{s} ds$ Ito integals are Mg's QV of JTs dWs is



 $(3) \partial_{\lambda,1}^2 = e_{mp}() \cdot 1$ $(4) d [M,M]_{1} = W_{1}^{2} dt$ $\Rightarrow dE(t) = 2f dt + 2f dM + \frac{1}{2}2f d[M,M]$ $= \mathcal{E}(t) \left[- \left\{ (t, w_t) \right\} dt + \mathcal{E}(t) w_t dw_t + \right]$ $+\frac{1}{2} \mathcal{E}(t) W_{1}^{2} dt$

 $= \varepsilon(t) \left[- \left\{ (t, W_t) + \frac{1}{2} W_t^2 \right] dt + \varepsilon(t) W_t dW_t \right]$

Problem 7.5. Suppose $\sigma = \sigma_t$ is a deterministic (i.e. non-random) process, and M is a martingale such that $d[M, M]_t = \sigma_t^2 dt$. (1) Given $\lambda, s, t \in \mathbb{R}$ with $0 \leq s < t$ compute $Ee^{\lambda M_t}$ and $E_s e^{\lambda M_t - M_s}$ (2) If $r \leq s$ compute $E \exp(\lambda M_r + \mu(M_t - M_s))$. (3) What is the joint distribution of $(M_r, M_t - M_s)$? (4) (Lévy's criterion) If $d[M,M]_t = dt$, then show that <u>M</u> is a standard Brownian motion.

"lample $E e^{\lambda M_{t}}$ $\pm \left(M_{q} \right) \rightarrow \left(M_{r} \right)$

 $f(t, x) = e^{\lambda x}$ $\partial_{t} f = 0$, $\partial_{t} f = \lambda e^{\lambda x}$, $\partial_{x}^{2} f = \lambda e^{\lambda x}$ $d[M,M] = v_t^2 dt$ Let 19(2) = E e $d\left(e^{\lambda M_{t}}\right) \stackrel{\text{I+o}}{=} 2_{t} dt + 2_{t} dM + \frac{1}{2} \partial_{x}^{2} f A[M,M]$ $= 0 + \lambda e^{\lambda M_{t}} dM + \frac{1}{2} \lambda e^{\lambda M_{t}} \tau_{t}^{2} dt$

 $\Rightarrow \underbrace{e^{\lambda M_{t}}}_{1} - \underbrace{e^{\lambda M_{0}}}_{1} = \lambda \int_{0}^{t} e^{\lambda M_{s}} dM_{s} + \frac{1}{2} \lambda^{z} \int_{0}^{t} e^{\lambda M_{s}} r_{s}^{z} ds$ Mg

 $Q(t) = E e^{\lambda M_t}$ $\Rightarrow Q(t) = 1 + 0 + \frac{1}{2}\lambda^{2} \int Q(s) \tau_{s}^{2} ds$ $\Rightarrow \varphi'(t) = \frac{\lambda^2}{2} \varphi(t) r_t^2$

 $\implies \ln \left(q(t) - \ln \left(q(0) \right) \right) = \frac{\chi^2}{2} \int_{0}^{t} \nabla_{s}^{2} ds$ $\Rightarrow \varphi(t) = \varphi(0) exp\left(\frac{\lambda^2}{2} \int_{0}^{1} \tau_{\zeta}^2 ds\right)$

 $1 \cdot exp\left(\frac{\lambda^2}{2} \int_{0}^{t} \nabla_{s}^{2} ds\right)$ $\Rightarrow E e^{\lambda M_t} =$ $M_{g} = \frac{1}{2} N(O, \int \tau_{g}^{2} ds)$ M $\gg M_{t} \sim N(0, \int_{0}^{t} T_{c}^{2} ds)$

Use the same truck to compute $E_s e^{\lambda(M_t - M_s)}$ $E_{s}e^{\lambda M_{b}} - e^{\lambda M_{s}} = \lambda E_{s}\int_{s}^{t}e^{\lambda M_{b}}dM_{b} + \frac{1}{2}\lambda E_{s}\int_{s}^{t}e^{\lambda M_{b}}de^{\alpha t}$ $\oint_{S} f = e^{\lambda(M_{t} - M_{s})} = e_{r} \left(\frac{\lambda^{2}}{2} \int_{r}^{t} \nabla_{r}^{2} dr \right) - M_{s} f = e_{r} \left(\frac{\lambda^{2}}{2} \int_{r}^{t} \nabla_{r}^{2} dr \right) - M_{s} f = e_{r} \left(\frac{\lambda^{2}}{2} \int_{r}^{t} \nabla_{r}^{2} dr \right) - M_{s} f = e_{r} \left(\frac{\lambda^{2}}{2} \int_{r}^{t} \nabla_{r}^{2} dr \right) - M_{s} f = e_{r} \left(\frac{\lambda^{2}}{2} \int_{r}^{t} \nabla_{r}^{2} dr \right) - M_{s} f = e_{r} \left(\frac{\lambda^{2}}{2} \int_{r}^{t} \nabla_{r}^{2} dr \right) - M_{s} f = e_{r} \left(\frac{\lambda^{2}}{2} \int_{r}^{t} \nabla_{r}^{2} dr \right) - M_{s} f = e_{r} \left(\frac{\lambda^{2}}{2} \int_{r}^{t} \nabla_{r}^{2} dr \right) - M_{s} f = e_{r} \left(\frac{\lambda^{2}}{2} \int_{r}^{t} \nabla_{r}^{2} dr \right) - M_{s} f = e_{r} \left(\frac{\lambda^{2}}{2} \int_{r}^{t} \nabla_{r}^{2} dr \right) - M_{s} f = e_{r} \left(\frac{\lambda^{2}}{2} \int_{r}^{t} \nabla_{r}^{2} dr \right) - M_{s} f = e_{r} \left(\frac{\lambda^{2}}{2} \int_{r}^{t} \nabla_{r}^{2} dr \right) - M_{s} f = e_{r} \left(\frac{\lambda^{2}}{2} \int_{r}^{t} \nabla_{r}^{2} dr \right) - M_{s} f = e_{r} \left(\frac{\lambda^{2}}{2} \int_{r}^{t} \nabla_{r}^{2} dr \right) - M_{s} f = e_{r} \left(\frac{\lambda^{2}}{2} \int_{r}^{t} \nabla_{r}^{2} dr \right) - M_{s} f = e_{r} \left(\frac{\lambda^{2}}{2} \int_{r}^{t} \nabla_{r}^{2} dr \right) - M_{s} f = e_{r} \left(\frac{\lambda^{2}}{2} \int_{r}^{t} \nabla_{r}^{2} dr \right) - M_{s} f = e_{r} \left(\frac{\lambda^{2}}{2} \int_{r}^{t} \nabla_{r}^{2} dr \right) - M_{s} f = e_{r} \left(\frac{\lambda^{2}}{2} \int_{r}^{t} \nabla_{r}^{2} dr \right) - M_{s} f = e_{r} \left(\frac{\lambda^{2}}{2} \int_{r}^{t} \nabla_{r}^{2} dr \right) - M_{s} \left$

(swells like M_-M_ shall be ind of F_) Wole: X & Y are ind $\Longrightarrow E(e^{\lambda X} + \mu Y) = Ee^{\lambda X} Ee^{\mu Y}$ fradut of MGF $(for all \lambda, \mu)$

Lets comparte
$$E\left(e^{\lambda M_{r}} + \mu\left(\frac{M_{t}}{t} - \frac{M_{s}}{s}\right)\right)$$
 $\left(r \leq s \leq t\right)$
 $= E E_{c}\left(\frac{1}{2}e^{\lambda M_{r}} + \mu\left(\frac{M_{t}}{t} - \frac{M_{s}}{s}\right)\right)$
 $= E E_{c}\left(e^{\lambda M_{r}} + \mu\left(\frac{M_{t}}{t} - \frac{M_{s}}{s}\right)\right) - E\left(e^{\lambda M_{r}} + E_{s}e^{\mu\left(\frac{M_{t}}{t} - \frac{M_{s}}{s}\right)}\right)$

 $= E\left(e^{\lambda M_{r}} e^{\frac{\lambda^{2}}{2}\int_{u}^{t} \tau_{u}^{2} du}\right)$ $= \rho_{2}^{2} \int \overline{\eta}_{n}^{2} dn + h_{2}^{2} \int \overline{\eta}_{n}^{2} dn$

= MGF of a 2D nond mean D $k lowige <math>\left(\int_{0}^{T} \nabla_{n}^{2} dn \int_{0}^{T} \nabla_{n}^{2} dn \right)$

 \Rightarrow $M_{t} - M_{s}$ is ind of M_{r} . for $M_t - M_s \sim N(0, t-s) = M_i c$ $k \text{ ind of } f_s$ they y's (man)

Problem 7.6. Define the process X, Y by

$$\longrightarrow X = \int_0^t \underline{s} \, dW_s \,, \quad Y = \int_0^t \underline{W}_s \, ds \,.$$

Find a formula for $\boldsymbol{E}X_t^n$ and $\boldsymbol{E}Y_t^n$ for any $n \in \mathbb{N}$.

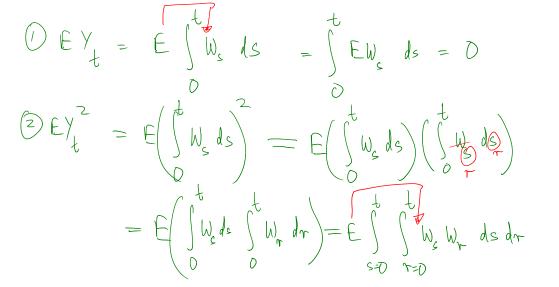
Claim: Both X & Y are Normal's

$$X_{t} = \lim_{t \to \infty} Z(t_{t}(W_{t} - W_{t})) \longrightarrow X \text{ is Word.}$$

hinen contriput of Words!

To find EX just find EX, & EXt k me the fourta for mails of Normal RV's. $EX_t = E \int s dW_s = 0$ $E \chi_{\ell}^{2} = E \left(\int_{0}^{t} \frac{s}{s} dW_{s}\right)^{2} \xrightarrow{\text{Ito Isom}} E \int_{0}^{t} \frac{s^{2}}{s^{2}} ds = \frac{t^{3}}{3}.$

Some for Y of $Y_{t} = \int_{0}^{t} W_{s} ds = \lim_{\substack{N \neq 0 \\ N \neq 0}} Z_{t} W_{s} (t_{t} - t_{t})$ Normal Not woodow. hiver coub of Normal ⇒ Y is Normel.



$$= \int_{SD} f E(W_{S}W_{T}) ds dr$$

$$= \int_{SD} f E(W_{S}W_{T}) ds dr & \text{integrate}_{b}$$

$$= \int_{SD} f (SAT) ds dr & \text{integrate}_{b}$$
het Meen & Variae & we MGF of Nound to find all monts!