Last time

Definition 6.24. A *semi-martingale* is a process of the form $X = X_0 + B + M$ where:

- ▷ X_0 is $\mathcal{F}_0$-measurable (typically X_0 is constant).
- ▷ B is an adapted process with finite first variation. (bdd variation)
- ▷ M is a martingale.

Definition 6.25. An *Itô-process* is a semi-martingale $X = X_0 + B + M$, where:

- ▷ $B_t = \int_0^t b_s \, ds$, with $\int_0^t |b_s| \, ds < \infty$ (Rieman Int)
- ▷ $M_t = \int_0^t \sigma_s \, dW_s$, with $\int_0^t |\sigma_s|^2 \, ds < \infty$ (Itô int)

dB dM

Remark 6.26. Short hand notation for Itô processes: $dX_t = b_t \, dt + \sigma_t \, dW_t$.

Remark 6.27. Expressing $X = X_0 + B + M$ (or $dX = b \, dt + \sigma \, dW$) is called the *semi-martingale decomposition* or the *Itô decomposition* of X.

Theorem 6.28 (Itô formula)**.** *If $f \in C^{1,2}$, then*

$$df(t, X_t) = \partial_t f(t, X_t)\, dt + \partial_x f(t, X_t)\, dX_t + \frac{1}{2}\partial_x^2 f(t, X_t)\, d[X, X]_t$$

Remark 6.29. This is the main tool we will use going forward. We will return and study it thoroughly after understanding all the notions involved.

More today

Proposition 6.30. *If $X = X_0 + B + M$, then $[X, X] = [M, M]$.*

(M, B are both cts)

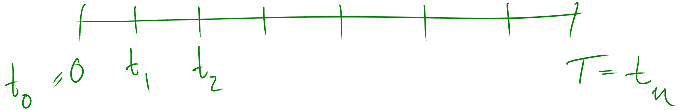
Note: $dM = \cancel{dW}\ \sigma\, dW$ then $M_t = \cancel{M_0}^{0} + \int_0^t \sigma_s\, dW_s$.

Know $[M, M]_t = \int_0^t \sigma_s^2\, ds$

($\Leftrightarrow$ $d[M, M]_t = \sigma_t^2\, dt$ (short hand notation).)

Alt Notation : $\quad d[M,M]_t = dM_t \, dM_t$

Intuition :



$$[X,X]_T = \lim_{\|P\| \to 0} \sum (\Delta_i X)^2 \qquad (\Delta_i X = X_{t_{i+1}} - X_{t_i})$$

$$= \lim_{\|P\| \to 0} \sum (\Delta_i B + \Delta_i M)^2$$

$$= \lim_{\|P\|\to 0} \sum \underbrace{(\Delta_i B)^2}_{\xrightarrow[\|P\|\to 0]{\text{IOU}}\ 0} + \underbrace{(\Delta_i M)^2}_{\xrightarrow{\|P\|\to 0}\ [M,M]_T} + \underbrace{2(\Delta_i B)(\Delta_i M)}_{\xrightarrow[\|P\|\to 0]{\text{IOU}}\ 0}$$

① Compute $\displaystyle \lim_{\|P\|\to 0} \sum (\Delta_i B)^2 \le \lim_{\|P\|\to 0} \Big(\underbrace{\max_i |\Delta_i B|}\ \underbrace{\sum |\Delta_i B|} \Big)$

$$\lim_{\|P\|\to 0} \max_i |B_{t_{i+1}} - B_{t_i}| = 0$$

$(\because B \text{ is cts})$

$\|P\| \to 0$

$V_{[0,T]}^{B}$

$(1^{st}\text{ Var } B) < \infty.$

$= 0$

$$② \quad \lim_{\|P\|\to 0} \cancel{X} \sum (\Delta_i B)(\Delta_i M) \overset{\text{Cauchy Schwatz}}{\le} \lim_{\|P\|\to 0} \underbrace{\underbrace{\left(\sum |\Delta_i B|^2\right)^{1/2}}_{\downarrow \atop 0} \underbrace{\left(\sum |\Delta_i M|^2\right)^{1/2}}_{\downarrow \|P\|\to 0 \atop [M,M]}}_{0}$$

$$\left(\underbrace{\sum x_i y_i}_{x \cdot y} \le \underbrace{\left(\sum x_i^2\right)^{1/2}}_{\|x\|} \underbrace{\left(\sum y_i^2\right)^{1/2}}_{\|y\|} . \right)$$

Proposition 6.31 (Uniqueness). *The Itô decomposition is unique. That is, if* $X = X_0 + B + M = Y_0 + C + N$, *with:*

- ▷ B, C *bounded variation,* $B_0 = C_0 = 0$
- ▷ M, N *martingale,* $M_0 = N_0 = 0$.

Then $X_0 = Y_0$, $B = C$ *and* $M = N$.

Check: ① At $t=0$, $M_0 = N_0 = 0$, $B_0 = C_0 = 0$

$\Rightarrow X_0 = Y_0$.

② Know $B + M = C + N$

$\Rightarrow B - C = \underbrace{N - M}$

Finite 1^{st} var $\qquad$ Mg

(B.V.).

Know $\quad E(N_t - M_t)^2 = E[N-M, N-M]_t$

$$= E[B-C, B-C]_t = 0$$

$\Rightarrow N = M \quad \& \quad B = C$

Corollary 6.32. *Let* $dX_t = \boxed{b_t\,dt} + \sigma_t\,dW_t$ *with* $\boldsymbol{E}\int_0^t b_s\,ds < \infty$ *and* $\boldsymbol{E}\int_0^t \sigma_s^2\,ds < \infty$. *Then* X *is a martingale if and only if* $b = 0$.

(If X was a mg, then

$$X_t = X_0 + \overset{BV}{0} + \overset{Mg}{(X_t - X_0)}$$

$$= X_0 + \int_0^t b_s\,ds + \int_0^t \sigma_s\,dW_s$$

Uniqueness $\Rightarrow \int_0^t b_s\,ds = 0 \ \forall t$ & $X_t = X_0 + \int_0^t \sigma_s\,dW_s$ $(\Rightarrow b = 0)$

Definition 6.33. If $dX = b\,dt + \sigma\,dW$, define $\int_0^T D_t\,dX_t = \int_0^T D_t b_t\,dt + \int_0^T D_t \sigma_t\,dW_t$.

Remark 6.34. Note $\int_0^T D_t b_t\,dt$ is a *Riemann integral*, and $\int_0^T D_t \sigma_t\,dW_t$ is a *Itô integral*.

6.6. Itô's formula.

Remark 6.35. If f and X are differentiable, then

$$df(t, X_t) = \partial_t f(t, X_t)\, dt + \partial_x f(t, X_t)\, dX_t$$

Chain Rule: $f = f(t, x)$ diff.

Want $x = x(t)$ some diff fn.

What is $\frac{d}{dt}\left[f(t, x(t)) \right] \underset{\text{Chain Rule}}{=} \partial_t f(t, x(t)) + \partial_x f(t, x(t)) \frac{dx}{dt}$

$x(t) \longrightarrow X_t$

$Y_t = f(t, X_t) \longrightarrow$ some process

$$dY_t = \left(\frac{d}{dt} f(t, X_t)\right) dt$$

$$= \partial_t f(t, X_t)\, dt + \partial_x f(t, X_t) \frac{dX_t}{\cancel{dt}} \cancel{dt}$$

$$d(f(t, X_t)) = \underline{\partial_t f(t, X_t)}\, dt + \partial_x f(t, X_t)\, dX_t$$

ONLY WORKS IF X is a diff fn of t.

All (non-constant) Mg's are NOT diff fn's of t.

Itô-Doeblin

Theorem (Itô's formula, Theorem 6.28). *If $f \in C^{1,2}$, then*

$$df(t, X_t) = \partial_t f(t, X_t)\, dt + \partial_x f(t, X_t)\, dX_t + \frac{1}{2}\partial_x^2 f(t, X_t)\, d[X, X]_t$$

Remark 6.36. If $dX_t = b_t\, dt + \sigma_t\, dW_t$ then

$$\cancel{df(t, X_t) = \Big(\partial_t f(t, X_t) + b_t + \frac{1}{2}\sigma_t^2\Big)\, dt + \partial_x f(t, X_t)\sigma_t\, dW_t.}$$

$f \in C^{1,2} \longrightarrow f = f(t, x)$

① f is diff in t

② f is twice diff in x.

$Y_t = f(t, X_t) \longleftarrow$ new process.

Q: $dY_t = ?$

Note $dX = b\,dt + \sigma\,dW$

$$d f(t, X_t) = \partial_t f(t, X_t)\,dt + \partial_x f(t, X_t)\,dX + \frac{1}{2}\partial_x^2 f(t, X_t)\,d[X,X]$$

$$= \partial_t f\,dt + \partial_x f\,(b\,dt + \sigma\,dW) + \frac{1}{2}\partial_x^2 f\,\sigma^2\,dt$$

$$= \left(\partial_t f + \partial_x f \cdot b + \frac{1}{2}\partial_x^2 f\,\sigma^2\right) dt + \partial_x f\,\sigma\,dW$$

Intuition behind Itô's formula?

Simple case : $f(t, x) = f(x)$ (ind of t).

$$X_t = W_t.$$

Itô : $d f(X_t) = d f(W_t) = f'(W_t)\, dt + \frac{1}{2} f''(W_t)\, dt$

Will show

$$t_0 = 0 \quad t_1 \quad \cdots \quad t_{n-1} \quad T = t_n$$

$$f(W_T) - f(W_0) = \sum \Delta_i f(W) = \sum f(W_{t_{i+1}}) - f(W_{t_i})$$

Taylor Exp

$$= \underbrace{\sum f'(W_{t_i})(W_{t_{i+1}} - W_{t_i})}_{1} + \underbrace{\frac{1}{2} \sum f''(W_{t_i})(W_{t_{i+1}} - W_{t_i})^2}_{2}$$

$$+ \underbrace{\text{small terms}}_{} \longrightarrow 0$$

a

$$\lim_{\|P\|\to 0} ① = \lim_{\|P\|\to 0} \sum f'(W_{t_i})\, \Delta_i W \longrightarrow \int_0^T f'(W_t)\, dW_t$$

$$\lim_{\|P\|\to 0} ② = \lim_{\|P\|\to 0} \frac{1}{2} \sum f''(W_{t_i}) (\Delta_i W)^2$$

$$= \lim_{\|P\|\to 0} \frac{1}{2} \sum f''(W_{t_i}) (t_{i+1} - t_i) \xrightarrow{\|P\|\to 0} \frac{1}{2}\int_0^T f''(W_t)\, dt$$

$''$

$$+ \lim_{\|P\| \to 0} \frac{1}{2} \sum \underbrace{f''(W_{t_i}) \cdot \left((\Delta_i W)^2 - (t_{i+1} - t_i) \right)}$$

NTS $\longrightarrow 0$

Note : $(\Delta_i W)^2 - (t_{i+1} - t_i) \sim \underbrace{\left(N(0, t_{i+1} - t_i)^2 - (t_{i+1} - t_i) \right)}$

① Mean 0.

② Variance $2(t_{i+1} - t_i)^2$

Guess : $\frac{1}{2}\sum f''(W_{t_i})\left((\Delta_i W)^2-(t_{i+1}-t_i)\right)$

mean 0 & variance $\underbrace{\sum f''(W_{t_i})^2\cdot 2(t_{i+1}-t_i)^2}$ $\xrightarrow[|P|\to 0]{} 0$

Example 6.37. Find the quadratic variation of W_t^2.

Q: Q4 What is QV of $2W_t$? $\longrightarrow$ Guess $4t$. ✓

What is QV of tW_t? $\longrightarrow$ Guess t^3 ✓

What is QV of $W_t W_t$ $\longrightarrow$ Guess $W_t^2 t$

Hopelessly Wrong !!

QV of W:

Let $X_t = W_t$.

Let $f(t, x) = x^2$

Then $W_t^2 = f(t, W_t)$.

By Ito: $d(W_t^2) = \partial_t f(t, X_t)\, dt + \partial_x f(t, X_t)\, dX_t + \frac{1}{2} \partial_x^2 f(t, X_t)\, \underbrace{d[X, X]}$.

$$\textcircled{1}\ \partial_t f = 0 \qquad = 0 + 2X_t\, dX_t$$

$$\textcircled{2}\ \partial_x f = 2x \qquad + \frac{1}{2}\cdot 2\cdot dt$$

$$\textcircled{3}\ \partial_x^2 f = 2 \qquad = 2W_t\, dW_t + dt$$

$$\textcircled{4}\ d[X,X] = dt.$$

$$\Rightarrow d(W_t^2) = 2W_t\, dW_t + \underline{dt}$$

$$\Rightarrow d\left[W^2, W^2\right]_t = 4W_t^2\,dt + 0$$

$$\Rightarrow \left[W, W\right]_T = \int_0^T 4W_t^2\,dt$$

Example 6.38. Find $\int_0^t W_s \, dW_s$. ← Shulan's Recitation.

Example 6.39. Let $M_t = W_t$, and $N_t = W_t^2 - t$.

- ▷ We know M, N are martingales.
- ▷ Is MN a martingale?

Is $W_t(W_t^2 - t)$ a mg?

$\iff$ Is $W_t^3 - W_t t$ a mg?

Option 1: Find $E_s(W_t^3 - W_t t)$ by Writing $W_t = W_t - W_s + W_s$ & ind lemma

Option 2: Itô's formula:

$$Y_t = f(t, W_t), \quad \text{where } f(t,x) = x^3 - xt.$$

Compute $dY = \text{Itô} = (\sim\sim)\,dt + (\sim\sim)\,dW$

Y is a mg if and only if the dt term vanishes.

① $\partial_t f = -x$

② $\partial_x f = 3x^2 - t$

③ $\partial_x^2 f = 6x$

④ $d[W,W]_t = dt$

(Replace x with W_t in)

$$dY = d\,f(t, \underbrace{W_t})$$

$$= \partial_t f \, dt + \partial_x f \, dW + \frac{1}{2}\partial_x^2 f \, dt$$

$$= -W_t \, dt + (3W_t^2 - t)\, dW + \frac{1}{2}\cdot 6 W_t \, dt$$

$$= \underbrace{2W_t \, dt}_{\neq 0} + (3W_t^2 - t)\, dW$$

$\Rightarrow f(t, W_t)$ is NOT a mg.

Example 6.40. Let $X_t = t\sin(W_t)$. Is $X_t^2 - [X,X]_t$ a martingale?

Guess: IDK (X need not be a mg)

Ito: $f(t,x) = t\sin x$

$\partial_t f = \sin x$

$\partial_x f = t\cos x$

$\partial_x^2 f = -t\sin x$

$$dX_t = \partial_t f\, dt + \partial_x f\, dW + \frac{1}{2}\partial_x^2 f\, dt$$

$$= \sin(W_t)\, dt + t\cos W_t\, dW - \frac{1}{2} t\sin(W_t)\, dt$$

$$= \left(\underbrace{\left(1 - \frac{t}{2}\right) \sin W_t}_{} \right) dt + t \cos W_t \, dW.$$

$$\Rightarrow d[X,X]_t = \underbrace{t^2 \cos^2(W_t)\, dt}_{}$$

NTS Check $X^2 - [X,X]$ is a mg.

$$Y = X^2 - [X,X] \Rightarrow dY = 2X\,dX + \frac{1}{2} \cdot 2 \cdot d[X,X] - d[X,X]$$

$$= 2X\,dX$$

$$= 2X\left(\left(1-\frac{t}{2}\right)\sin W_t\right)dt \Rightarrow \neq 0$$

$$+ 2X\,t\cos W_t\,dW$$

$\Rightarrow X^2 - [X,X]$ is not a mg!