Def. A Brownian motion W is a continuous process s.t.

(1) $W_0 = 0$

(2) $W_t - W_s \sim N(0, t-s)$, $0 \le s < t$ $\qquad (W_t \sim N(0,t))$

(3) $W_t - W_s$ is independent of $\mathcal{F}_s$, $0 \le s < t$

Ex. Find the distribution of (W_s, W_t) $(0 < s < t)$

W_s and $W_t - W_s$ are independent , $W_s \sim N(0,s)$, $W_t - W_s \sim N(0, t-s)$

$$\begin{pmatrix} W_s \\ W_t - W_s \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} s & 0 \\ 0 & t-s \end{pmatrix} \right)$$

$$\begin{pmatrix} W_s \\ W_t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} W_s \\ W_t - W_s \end{pmatrix}$$

$W_s \sim N(0,s)$, $W_t \sim N(0,t)$, $\mathrm{Cov}(W_s, W_t) = \mathbb{E}(W_s W_t) - \underline{\mathbb{E} W_s}\, \underline{\mathbb{E} W_t}$

$$= \mathbb{E}(W_s(W_t - W_s + W_s))$$

$$= \mathbb{E}(W_s(W_t - W_s)) + \mathbb{E}(W_s)^2$$

$$= \mathbb{E} W_s \cdot \mathbb{E}(W_t - W_s) + \mathrm{Var}(W_s) = s$$

$$\Rightarrow \begin{pmatrix} W_s \\ W_t \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} s & s \\ s & t \end{pmatrix} \right).$$

For $s > 0$, $t > 0$, $\mathrm{Cov}(W_s, W_t) = \mathbb{E}(W_s W_t) = \min\{s, t\} = s \wedge t$.

Ex. $X_T = \int_0^T e^{W_t}\, dt$, $T \geq 0$ $\quad \left(X_T(\omega) = \int_0^T e^{W_t(\omega)}\, dt \right)$

Compute $\mathbb{E} X_T$, $\mathrm{Var}(X_T)$.

Recall: Moment Generating Function, $Z \sim N(0,1)$, $M_Z(t) = \mathbb{E} e^{tZ} = e^{\frac{t^2}{2}}$.

$$\mathbb{E} X_T = \mathbb{E}\left(\int_0^T e^{W_t}\, dt \right) = \int_0^T \mathbb{E} e^{W_t}\, dt = \int_0^T \underline{\mathbb{E} e^{\sqrt{t} Z}}\, dt = \int_0^T e^{\frac{t}{2}}\, dt = 2\left(e^{\frac{T}{2}} - 1\right).$$

$(W_t \sim N(0,t))$

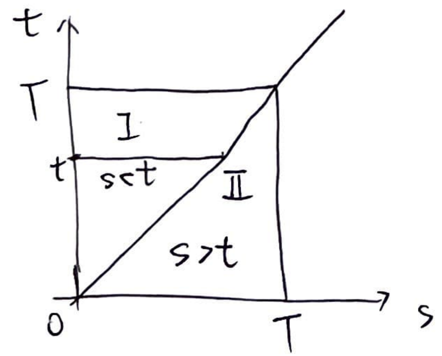
$$\mathbb{E} X_T^2 = \mathbb{E}\left(\int_0^T e^{W_t}\, dt \right)^2 = \mathbb{E}\left(\int_0^T e^{W_s}\, ds \cdot \int_0^T e^{W_t}\, dt \right) = \mathbb{E}\left(\int_0^T \int_0^T e^{W_s + W_t}\, ds\, dt \right)$$

$$= \int_0^T \int_0^T \mathbb{E} e^{W_s + W_t}\, ds\, dt$$

$\mathbb{E}(W_s + W_t) = 0$, $\mathrm{Var}(W_s + W_t) = \mathrm{Var}(W_s) + \mathrm{Var}(W_t) + 2\,\mathrm{Cov}(W_s, W_t) = s + t + 2(s \wedge t)$.

$W_s + W_t \sim N(0, s + t + 2(s \wedge t))$.

$$\mathbb{E} X_T^2 = \int_0^T \int_0^T \mathbb{E} e^{\sqrt{s + t + 2(s\wedge t)}\, Z}\, ds\, dt = \int_0^T \int_0^T e^{\frac{s}{2} + \frac{t}{2} + s \wedge t}\, ds\, dt$$



$$= \iint_{I} e^{\frac{3}{2}s + \frac{1}{2}t}\, ds\, dt + \iint_{II} e^{\frac{1}{2}s + \frac{3}{2}t}\, ds\, dt$$

$$= 2 \iint_{I} e^{\frac{3}{2}s + \frac{t}{2}}\, ds\, dt = 2 \int_0^T \int_0^t e^{\frac{3}{2}s + \frac{t}{2}}\, dt$$

$$\mathbb{E}X_T^2 = 2\int_0^T e^{\frac{t}{2}} \int_0^t e^{\frac{3}{2}s}\,ds\,dt = 2\int_0^T e^{\frac{t}{2}} \cdot \frac{2}{3}\left(e^{\frac{3}{2}t}-1\right)dt = \frac{2}{3}e^{2T} - \frac{8}{3}e^{\frac{T}{2}} + 2.$$

$$\mathrm{Var}(X_T) = \mathbb{E}X_T^2 - (\mathbb{E}X_T)^2 = \frac{2}{3}e^{2T} - 4e^{T} + \frac{16}{3}e^{\frac{T}{2}} - 2.$$

- Continuous time (C) BM W_t

  Discrete time (D) SRW $S_n = \xi_1 + \cdots + \xi_n$, ξ_i i.i.d. $\mathbb{P}(\xi_i = 1) = \mathbb{P}(\xi_i = -1) = \frac{1}{2}$.

- Quadratic Variation

(C) X_t is a continuous adapted process,

$$[X, X]_T = \lim_{\|P\| \to 0} \sum_{k=0}^{n-1} \left(X_{t_{k+1}} - X_{t_k}\right)^2, \quad 0 = t_0 < t_1 < \cdots < t_n = T.$$

$$[W, W]_T = T.$$

(D) X_n is an adapted process

$$[X, X]_N = \sum_{k=0}^{N-1} \left(X_{k+1} - X_k\right)^2.$$

$$[S, S]_n = \sum_{k=0}^{n-1} \left(S_{k+1} - S_k\right)^2 = \sum_{k=0}^{n-1} (\xi_{k+1})^2 = \sum_{k=0}^{n-1} 1 = n.$$

(C) If M_t is a continuous MRT, then

$M_t^2 - [M, M]_t$ is also a continuous MRT.

(D) If M_n is a ~~continuous~~ MRT, then

$M_n^2 - [M, M]_n$ is also a continuous MRT.

pf. (1) $M_n^2 - \sum_{k=0}^{n-1} (M_{k+1} - M_k)^2$ is $\mathcal{F}_n$-measurable.

(2) $\mathbb{E}_n\left(M_{n+1}^2 - [M, M]_{n+1}\right) = \mathbb{E}_n\left(M_{n+1}^2 - \sum_{k=0}^{n} (M_{k+1} - M_k)^2\right)$

$= \mathbb{E}_n\left(M_{n+1}^2 - \underbrace{\sum_{k=0}^{n-1} (M_{k+1} - M_k)^2}_{=[M,M]_n} - (M_{n+1} - M_n)^2\right)$

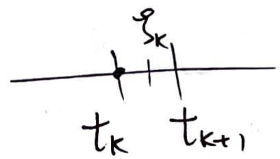
$= \mathbb{E}_n\left(M_{n+1}^2 - (M_{n+1} - M_n)^2\right) - [M, M]_n$

$= \mathbb{E}_n\left(2M_{n+1}M_n - M_n^2\right) = 2M_n \mathbb{E}_n M_{n+1} - M_n^2 = 2M_n^2 - M_n^2 = M_n^2$

- Itô Integral

(C) D_t is an adapted process

$$I_T = \int_0^T D_t \, dW_t = \lim_{\|P\|\to 0} \left[\sum_{k=0}^{n-1} D_{t_k} \left(W_{t_{k+1}} - W_{t_k} \right) + D_{t_n} \left(W_T - W_{t_n} \right) \right], \quad T \in (t_n, t_{n+1})$$



I_t is a martingale.

(D) D_n is an adapted process

$$I_N = \sum_{k=0}^{N-1} D_k \Delta S_k = \sum_{k=0}^{N-1} D_k \left(S_{k+1} - S_k \right)$$

I_n is a martingale.

pf. (1) I_n is adapted ✓

(2) $$\mathbb{E}_n I_{n+1} = \mathbb{E}_n \left(\sum_{k=0}^{n} D_k (S_{k+1} - S_k) \right) = \mathbb{E}_n \Big(\underbrace{\sum_{k=0}^{n-1} D_k (S_{k+1} - S_k)}_{I_n} + D_n (S_{n+1} - S_n) \Big)$$

$$= I_n + \mathbb{E}_n \left(D_n (S_{n+1} - S_n) \right) = I_n + D_n \underbrace{\mathbb{E}_n (S_{n+1} - S_n)}_{=0} = I_n .$$

(C) $I_T = \int_0^T D_t \, dW_t$

$$[I, I]_T = \int_0^T D_t^2 \, dt \quad \left(= \int_0^T D_t^2 \underbrace{(dW_t)^2}_{= dt} \right)$$

(D) $I_N = \sum_{k=0}^{N-1} D_k (S_{k+1} - S_k)$

$$[I, I]_N = \sum_{k=0}^{N-1} (I_{k+1} - I_k)^2 = \sum_{k=0}^{N-1} D_k^2 (S_{k+1} - S_k)^2 = \sum_{k=0}^{N-1} D_k^2 (\xi_{k+1})^2$$

$$= \sum_{k=0}^{N-1} D_k^2 \left(= \sum_{k=0}^{N-1} D_k^2 \left((k+1) - k \right) \right)$$

Def. A semi-martingale is a process of the form $X_t = X_0 + B_t + M_t$, where

(1) Typically X_0 is a constant.

(2) B_t is an adapted process with finite first variation (BV bounded variation)

$$\left([B, B]_T \cancel{\equiv \sum_{k=0}} = \lim_{\|P\| \to 0} \sum_{k=0}^{n-1} (B_{t_{k+1}} - B_k)^2 = \lim_{\|P\| \to 0} \sum_{k=0}^{n-1} |B_{t_{k+1}} - B_{t_k}| \cdot |B_{t_{k+1}} - B_{t_k}| \right.$$

$$\le \lim_{\|P\| \to 0} \underbrace{\sum_{k=0}^{n-1} |B_{t_{k+1}} - B_{t_k}|}_{\downarrow \; V_{[0,T]}(B) < \infty} \cdot \underbrace{\overset{\max |\Delta B_{t_k}|}{\cancel{\|P\|}}}_{\downarrow \; 0} = 0$$

~~$\|P\| = \max_{0 \le k \le n-1} |B_{t_{k+1}} - B_{t_k}|$~~

$\|P\| \to 0 \Rightarrow \max_k |\Delta B_{t_k}| \to 0$

(3) M_t is a martingale.

Def. An Itô process X is a semi-martingale, where

$$X_t = X_0 + \int_0^t b_s\, ds + \int_0^t \sigma_s\, dW_s, \quad \mathbb{E}\int_0^t |b_s|\, ds < \infty, \quad \mathbb{E}\int_0^t |\sigma_s|^2\, ds < \infty$$

Equivalently, $dX_t = b_t\, dt + \sigma_t\, dW_t$.

- $X_t = X_0 + B_t + M_t$ — semi-martingale.

  $[X, X]_t = [M, M]_t$

- Itô formula

  $f = f(t,x) : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, $f \in C^{1,2}$ ($\partial_t f$, $\partial_x f$, $\partial_x^2 f$ are continuous)

  $$df(t, X_t) = \partial_t f(t, X_t)\, dt + \partial_x f(t, X_t)\, \underline{dX_t} + \frac{1}{2}\partial_x^2 f(t, X_t)\, d[X,X]_t.$$

  $$f(t, X_t) = f(0, X_0) + \int_0^t \partial_t f(s, X_s)\, ds + \int_0^t \partial_x f(s, X_s)\, \underline{dX_s} + \frac{1}{2}\int_0^t \partial_x^2 f(s, X_s)\, d[X,X]_s.$$

  Suppose X is an Itô process

  $$X_t = X_0 + \underbrace{\int_0^t b_s\, ds}_{B_t} + \underbrace{\int_0^t \sigma_s\, dW_s}_{M_t}$$

  $\underline{dX_t} = b_t\, dt + \sigma_t\, dW_t$, $[X,X]_t = [M,M]_t = \int_0^t \sigma_s^2\, ds$, $\underline{d[X,X]_t} = \sigma_t^2\, dt$

$$df(t, X_t) = \partial_t f(t, X_t)\,dt + \partial_x f(t, X_t)\left(b_t\,dt + \sigma_t\,dW_t\right) + \frac{1}{2}\partial_x^2 f(t, X_t)\,\sigma_t^2\,dt$$

$$= \left(\partial_t f(t, X_t) + b_t\,\partial_x f(t, X_t) + \frac{1}{2}\sigma_t^2\,\partial_x^2 f(t, X_t)\right)dt + \sigma_t\,\partial_x f(t, X_t)\,dW_t$$

Ex. What is $\int_0^t W_s\,dW_s$?

$X_t = W_t^2 - t$ (X is a martingale)

$f(t, x) = x^2 - t$, $\partial_t f(t, x) = -1$, $\partial_x f(t, x) = 2x$, $\partial_x^2 f(t, x) = 2$.

$\underline{X_t = f(t, W_t)}$

$$dX_t = \partial_t f(t, W_t)\,dt + \partial_x f(t, W_t)\,dW_t + \frac{1}{2}\partial_x^2 f(t, W_t)\,dt$$

$$= \cancel{-dt} + 2W_t\,dW_t + \cancel{\frac{1}{2}\cdot 2\,dt} \qquad = 2W_t\,dW_t$$

$$X_t - X_0 = 2\int_0^t W_s\,dW_s \Rightarrow \int_0^t W_s\,dW_s = \frac{1}{2}X_t = \frac{1}{2}(W_t^2 - t).$$

Ex. $M_t = e^{\lambda W_t - \frac{1}{2}\lambda^2 t}$ (From HW2, M is a MRT). Find dM_t.

$f(t, x) = e^{\lambda x - \frac{1}{2}\lambda^2 t}$, $\partial_t f(t,x) = -\frac{1}{2}\lambda^2 f(t,x)$, $\partial_x f(t,x) = \lambda f(t,x)$, $\partial_x^2 f(t,x) = \lambda^2 f(t,x)$.

$$dM_t = df(t, W_t) = \partial_t f(t, W_t)\,dt + \partial_x f(t, W_t)\,dW_t + \frac{1}{2}\partial_x^2 f(t, W_t)\,dt$$

$$= \cancel{-\frac{1}{2}\lambda^2 M_t\,dt} + \lambda M_t\,dW_t + \cancel{\frac{1}{2}\lambda^2 M_t\,dt} \qquad = \lambda M_t\,dW_t.$$

- If $f(t, W_t)$ is a MRT, then $df(t, W_t)$ has no dt term.

$$f(t, W_t) = f(0,0) + \int_0^t (\cdots)\, dW_s.$$

Ex. $X_t = e^{W_t}$, Find $[X, X]_t$.

$f(t,x) = e^x$, $\partial_t f(t,x) = 0$, $\partial_x f(t,x) = \partial_x^2 f(t,x) = e^x$.

$X_t = f(t, W_t)$.

$$dX_t = df(t, W_t) = \partial_t f(t, W_t)\, dt + \partial_x f(t, W_t)\, dW_t + \frac{1}{2}\partial_x^2 f(t, W_t)\, dt$$

$$= 0 + \frac{1}{2} e^{W_t}\, dt + e^{W_t}\, dW_t.$$

$$X_t = X_0 + \underbrace{\frac{1}{2}\int_0^t e^{W_s}\, ds}_{B_t} + \underbrace{\int_0^t e^{W_s}\, dW_s}_{M_t}.$$

$$[X, X]_t = [M, M]_t = \int_0^t e^{2W_s}\, ds.$$