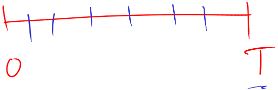
Last time:

1st Var: $V_{[0,T]} X = \lim_{\|P\| \to 0} \sum_{i=1}^{n} \underbrace{|\Delta_i X|}$



$P = \{0 = t_0 < t_1 < \dots t_n = T\}$

$\|P\| = \max_i \, t_{i+1} - t_i$

$\Delta_i X = X_{t_{i+1}} - X_{t_i}$

B.M does not have finite 1st Var.

$V_{[0,T]} W = +\infty$ (a.s).

Need $V_{[0,T]} X < \infty$

Quadratic Var:

$$[X,X]_T = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} (\Delta_i X)^2$$

in order to define Riemann Int

Saw $[W,W]_T = T$ (a.s.)

& $W_t^2 - [W,W]_t = W_t^2 - t$ is a mg.

Theorem 6.11. *Let M be a continuous martingale.*

(1) $\boldsymbol{E}M_t^2 < \infty$ *if and only if* $\boldsymbol{E}[M,M]_t < \infty$.
(2) *In this case* $M_t^2 - [M,M]_t$ *is a continuous martingale.*
(3) *Conversely, if* $M_t^2 - A_t$ *is a martingale for any continuous, increasing process* A *such that* $A_0 = 0$, *then we must have* $A_t = [M,M]_t$.

Remark 6.12. The optional problem on HW2 gives some intuition in discrete time.

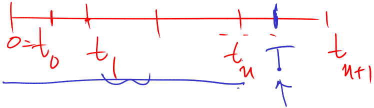
Remark 6.13. If X has finite first variation, then $|X_{t+\delta t} - X_t| \approx O(\delta t)$.

Remark 6.14. If X has finite quadratic variation, then $|X_{t+\delta t} - X_t| \approx O(\sqrt{\delta t}) \gg O(\delta t)$.

Intuition: Finite 1^{st} var $\longrightarrow$ "differentiable in time"

$\longrightarrow$ Finite (non-zero) QV $\longrightarrow$ Never diff

6.4. Itô Integrals.



- $D_t = D(t)$ some adapted process (position on an asset).
- $P = \{0 = t_0 < t_1 < \cdots\}$ increasing sequence of times.
- $\|P\| = \max_i (t_{i+1} - t_i)$, and $\Delta_i X = X_{t_{i+1}} - X_{t_i}$.
- W : standard Brownian motion.
- $I_P(T) \stackrel{\text{def}}{=} \sum_{i=0}^{n-1} D_{t_i} \Delta_i W + D_{t_n}(W_T - W_{t_n})$

if $T \in [t_n, t_{n+1})$

Definition 6.15. The *Itô Integral* of D with respect to Brownian motion is defined by

$$\int_0^T D_s\, dW_s = I_T = \int_0^T D_t \, dW_t = \lim_{\|P\| \to 0} I_P(T) .$$

Only works because D_t is an adapted process.

Remark 6.16. Suppose for simplicity $T = t_n$.

(1) Riemann integrals: $\lim_{\|P\|\to 0} \sum D_{\xi_i} \Delta_i W$ exists, for any $\xi_i \in [t_i, t_{i+1}]$.

(2) Itô integrals: Need $\xi_i = t_i$ for the limit to exist.

& Need D to be adapted.

Theorem 6.17. *If* $\boldsymbol{E}\int_0^T D_t^2\,dt < \infty$ ~~*a.s.*~~, *then:*

$$\left(I_T = \int_0^T D_t \, dW_t \right)$$

(1) $I_T = \lim_{\|P\|\to 0} I_P(T)$ *exists a.s., and* $\boldsymbol{E}I(T)^2 < \infty$.

(2) *The process* I_T *is a martingale:* $\boldsymbol{E}_s I_t = \boldsymbol{E}_s \int_0^t D_r\,dW_r = \int_0^s D_r\,dW_r = I_s$

(3) $[I,I]_T = \int_0^T D_t^2\,dt$ *a.s.*

(Note $\int_0^T D_t^2 \, dt$ is a std Riemann Int)

Remark 6.18. If we only had $\int_0^T D_t^2\,dt < \infty$ a.s., then $I(T) = \lim_{\|P\|\to 0} I_P(T)$ still exists, and is finite a.s. But it may not be a martingale (it's a *local martingale*).

NOTATION: $EX^2 = E(X^2)$ NOT $(EX)^2$

Corollary 6.19 (Itô isometry). $\boldsymbol{E}\Big(\int_0^T D_t\,dW_t\Big)^2 = \boldsymbol{E}\int_0^T D_t^2\,dt$ $= \int_0^T E D_t^2 \, dt$

Proof.

Re Note For Rieman Integrals,

$$E \int_0^T D_t^2 \, dt = \int_0^T E D_t^2 \, dt$$

NOT dW
Rieman Int

Intuition: $\mathbb{E}\int_0^T D_t^2\,dt$ (Riemann) $= \mathbb{E}\lim_{\|P\|\to 0} \sum D_{t_i}^2 (t_{i+1}-t_i)$

$$= \lim_{\|P\|\to 0} \mathbb{E}\sum D_{t_i}^2 (t_{i+1}-t_i)$$

$$= \lim_{\|P\|\to 0} \sum \left(\mathbb{E} D_{t_i}^2\right)(t_{i+1}-t_i)$$

$$= \int_0^T \left(\mathbb{E} D_t^2\right) dt$$

u

Pf of Itô isom (Assuming prop of Itô int):

Know $I_t = \int_0^t D_s \, dW_s$ is a mg

& $[I, I]_t = \int_0^t D_s^2 \, ds$

$\Rightarrow I_t^2 - [I, I]_t$ is a mg!

$\Rightarrow E\left(I_t^2 - [I, I]_t\right) = E\left(I_0^2 - [I, I]_0\right) = 0$

$$\Rightarrow \quad E I_t^2 = E [I, I]_t$$

$$\Rightarrow \quad E\left(\int_0^t D_s \, dW_s \right)^2 = E\left(\int_0^t D_s^2 \, ds \right)$$

//.

Intuition for Theorem 6.17 (2). Check $I_P(T)$ is a martingale.

$$I_P(T) = \sum_{i=0}^{n-1} D_{t_i} \Delta_i W + D_{t_n}(W_T - W_{t_n}) \quad \text{if } T \in [t_n, t_{n+1})$$

NTS $E_s I_P(t) = I_P(s)$

for simplicity suppose $s = t_m$ & $t = t_n$, $m \leq n$.

$$I_P(s) = I_P(t_m) = \sum_{i=0}^{m-1} D_{t_i} \Delta_i W \leftarrow$$

$$I_P(t) = \underline{I_P(t_n)} = \sum_{i=0}^{n-1} D_{t_i} \Delta_i W$$

$$\Rightarrow E_s(\quad) = E_{t_m}\left(\sum_{i=0}^{n-1} D_{t_i} \Delta_i W \right)$$

$$= \underbrace{E_{t_m}\left(\sum_{i=0}^{m-1} D_{t_i} \Delta_i W \right)}_{\mathcal{F}_{t_m}\text{-meas } (\because t_i < t_m)} + E_{t_m}\left(\sum_{i=m}^{n-1} D_{t_i} \left(W_{t_{i+1}} - W_{t_i} \right) \right)$$

$$= \underbrace{\sum_{i=0}^{m-1} D_{t_i} \Delta_i W}_{I_p(s)} + \sum_{i=m}^{n-1} E_{t_m} E_{t_i} \left[D_{t_i} \left(W_{t_{i+1}} - W_{t_i} \right) \right]$$

$$= I_p(s) + \sum_{i=m}^{n-1} E_{t_m} \left(D_{t_i} \, E_{t_i} \left(W_{t_{i+1}} - W_{t_i} \right) \right)$$

($\because$ D_{t_i} is $\mathcal{F}_{t_i}$-meas)

$$= I_p(s) + \sum_{i=m}^{n-1} E_{t_m}\Big(D_{t_i} \underbrace{E\big(W_{t_{i+1}} - W_{t_i}\big)}_{0} \Big)$$

$\Big(\because W_{t_{i+1}} - W_{t_i}$ is ind of $\mathcal{F}_i$

& $W_{t_{i+1}} - W_{t_i} \sim N(0, t_{i+1} - t_i)\Big)$

$$= I_p(s)$$

QED.

Intuition: Itô Isom: $\mathbb{E}\left(\int_0^T D_s \, dW_s\right)^2 = \mathbb{E}\int_0^T D_s^2 \, ds$

Let's check by hand: $\mathbb{E}\left(\sum_{i=0}^{n-1} D_{t_i} \Delta_i W\right)^2 = \underbrace{\mathbb{E}\sum_{i=0}^{n-1} D_{t_i}^2 (t_{i+1}-t_i)}_{1}$

Expand LHS:

$$\mathbb{E}\left(\sum_{i=0}^{n-1} D_{t_i} \Delta_i W\right)^2 = \mathbb{E}\left(\underline{\sum_{i=0}^{n-1} D_{t_i}^2 (\Delta_i W)^2} + \right.$$

$$\textcircled{1}$$

$$\underbrace{E\, 2 \sum_{j=0}^{n-1} \sum_{i=0}^{j-1} D_{t_i} \Delta_i W \; D_{t_j} \Delta_j W}_{\textcircled{2}}$$

$$\textcircled{1} = \sum_{i=0}^{n-1} E D_{t_i}^2 (\Delta_i W)^2 = \sum_{i=0}^{n-1} E \, E_{t_i}\left(D_{t_i}^2 \left(W_{t_{i+1}} - W_{t_i}\right)^2 \right)$$

$$= \sum_{i=0}^{n-1} E D_{t_i}^2 \; E\left(W_{t_{i+1}} - W_{t_i}\right)^2$$

$$= \sum_{i=0}^{n-1} E D_{t_i}^2 (t_{i+1} - t_i) = \text{Desired RHS.}$$

$$② \doteq 2 \sum_{j=0}^{n-1} \sum_{i=0}^{j-1} E\left(D_{t_i}(W_{t_{i+1}} - W_{t_i}) D_{t_j}(W_{t_{j+1}} - W_{t_j}) \right)$$

$$= 2 \sum_{j=0}^{n-1} \sum_{i=0}^{j-1} E\, E_{t_j}\left(D_{t_i}(W_{t_{i+1}} - W_{t_i})\, D_{t_j}(W_{t_{j+1}} - W_{t_j}) \right)$$

Note $i < j \Rightarrow D_{t_i}, W_{t_{i+1}}, W_{t_i}, D_{t_j}$ meas

$$= 2 \sum_{j=0}^{n-1} \sum_{i=0}^{j-1} E\left(D_{t_i} (W_{t_{i+1}} - W_{t_i}) D_{t_j} \underbrace{E(W_{t_{j+1}} - W_{t_j})}_{0} \right)$$

$$= 0$$

QED.

Proposition 6.20. *If $\alpha, \tilde{\alpha} \in \mathbb{R}$, $D, \tilde{D}$ adapted processes*

$$\int_0^T (\alpha D_s + \tilde{\alpha} \tilde{D}_s)\, dW_s = \alpha \int_0^T D_s\, dW_s + \tilde{\alpha} \int_0^T \tilde{D}_s\, dW_s$$

Proposition 6.21. $\underbrace{\int_0^{T_1} D_s\, dW_s + \int_{T_1}^{T_2} D_s\, dW_s}$ $= \int_0^{T_2} D_s\, dW_s$

Question 6.22. *If $\underbrace{D \geqslant 0}$, then must $\underbrace{\int_0^T D_t\, dW_t}$ $\geqslant 0$?* $\longleftarrow$ False!

Intuition: $\int_0^T (\alpha D_s + \tilde{\alpha} \tilde{D}_s)\, dW_s = \lim \sum (\alpha D_{t_i} + \tilde{\alpha} \tilde{D}_{t_i}) \Delta_i W$

$\lim \sum \alpha D_{t_i} \Delta_i W + \tilde{\alpha} \lim \sum \tilde{D}_{t_i} \Delta_i W$

6.5. **Semi-martingales and Itô Processes.**

Question 6.23. *What is* $\int_0^t W_s \, dW_s$*?*

Definition 6.24. A *semi-martingale* is a process of the form $X = X_0 + B + M$ where:

- ▷ X_0 is $\mathcal{F}_0$-measurable (typically X_0 is constant).
- ▷ B is an adapted process with finite first variation. (aka Bounded Variation)
- ▷ M is a martingale.

Definition 6.25. An *Itô-process* is a semi-martingale $X = X_0 + B + M$, where:

- ▷ $B_t = \int_0^t b_s \, ds$, with $\int_0^t |b_s| \, ds < \infty$ (Std Riemann int) $\to$ $dB_t = b_t \, dt$
- ▷ $M_t = \int_0^t \sigma_s \, dW_s$, with $\int_0^t |\sigma_s|^2 \, ds < \infty$ (Itô int) $\to$ $dM_t = \sigma_t \, dW_t$

Remark 6.26. Short hand notation for Itô processes: $dX_t = b_t \, dt + \sigma_t \, dW_t$.

Remark 6.27. Expressing $X = X_0 + B + M$ (or $dX = b \, dt + \sigma \, dW$) is called the *semi-martingale decomposition* or the *Itô decomposition* of X.

Theorem 6.28 (Itô formula)**.** *If* $f \in C^{1,2}$*, then*

$$df(t, X_t) = \partial_t f(t, X_t)\, dt + \partial_x f(t, X_t)\, dX_t + \boxed{\frac{1}{2}\partial_x^2 f(t, X_t)\, d[X, X]_t}$$

Remark 6.29*.* This is the main tool we will use going forward. We will return and study it thoroughly after understanding all the notions involved.

Proposition 6.30. *If* $X = X_0 + B + M$, *then* $[X, X] = [M, M]$.

Proposition 6.31 (Uniqueness)**.** *The Itô decomposition is unique. That is, if* $X = X_0 + B + M = Y_0 + C + N$*, with:*

- ▷ B, C *bounded variation,* $B_0 = C_0 = 0$
- ▷ M, N *martingale,* $M_0 = N_0 = 0$*.*

Then $X_0 = Y_0$*,* $B = C$ *and* $M = N$*.*