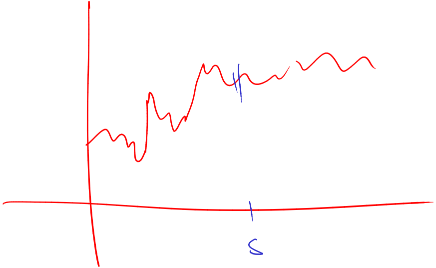
Last time: B. M.

$W_t \to$ B. M.

⓪ W is a c/s process

① $W_t \sim N(0, t-s)$

& ② $W_t - W_s$ is ind of $\mathcal{F}_s$



Definition 5.4. $\boldsymbol{E}_t X$ is the unique *random variable* such that:

(1) $\boldsymbol{E}_t X$ is $\mathcal{F}_t$-measurable.

(2) For every $A \in \mathcal{F}_t$, $\cancel{\frac{1}{P(A)}}\int_A \boldsymbol{E}_t X \, d\boldsymbol{P} = \cancel{\frac{1}{P(A)}}\int_A X \, d\boldsymbol{P}$

$\left(E(X \mid \mathcal{F}_t) = E_t X \right.$

$\left. \text{cond exp of } X \text{ given } \mathcal{F}_t \right)$

Remark 5.5. Choosing $A = \Omega$ implies $\boldsymbol{E}(\boldsymbol{E}_t X) = \boldsymbol{E} X$.

Proposition 5.6 (Useful properties of conditional expectation)**.**

(1) If $\alpha, \beta \in \mathbb{R}$ are constants, X, Y, random variables $\boldsymbol{E}_t(\alpha X + \alpha Y) = \alpha \boldsymbol{E}_t X + \beta \boldsymbol{E}_t Y$.

(2) If $X \geqslant 0$, then $\boldsymbol{E}_t X \geqslant 0$. Equality holds if and only if $X = 0$ almost surely.

(3) (Tower property) *If $0 \leqslant s \leqslant t$, then $\boldsymbol{E}_s(\boldsymbol{E}_t X) = \boldsymbol{E}_s X$.*

(4) If X is $\mathcal{F}_t$ measurable, and Y is any random variable, then $\boldsymbol{E}_t(XY) = X \boldsymbol{E}_t Y$.

(5) If X is $\mathcal{F}_t$ measurable, then $\boldsymbol{E}_t X = X$ (follows by choosing $Y = 1$ above).

(6) If Y is independent *of $\mathcal{F}_t$, then $\boldsymbol{E}_t Y = \boldsymbol{E} Y$.*

Remark 5.7. These properties are exactly the same as in discrete time.

Lemma 5.8 (Independence Lemma). *If X is $\mathcal{F}_t$ measurable, Y is independent of $\mathcal{F}_t$, and $f = f(x,y)\colon \mathbb{R}^2 \to \mathbb{R}$ is any function, then*

$$\boldsymbol{E}_t f(X,Y) = g(Y)\,, \qquad \textit{where} \qquad g(y) = \boldsymbol{E} f(X,y)\,.$$

~~*Remark* 5.9. If p_X is the PDF of X, then $\boldsymbol{E}_t f(X,Y) = \int_{\mathbb{R}} f(x,Y)\,p_X(x)\,dx$.~~

$p_Y \longrightarrow$ PDF of Y

$E_t f(X,Y) =$ "average Y & leave X alone"

$$= \int_{\mathbb{R}} f(X,y)\, p_Y(y)\, dy$$

5.4. **Martingales.**

Definition 5.10. An adapted process M is a martingale if for every $0 \leqslant s \leqslant t$, we have $\boldsymbol{E}_s M_t = M_s$.

Remark 5.11. As with discrete time, a martingale is a fair game: stopping based on information available today will not change your expected return.

Discrete time: M is a mg if $E_n M_{n+1} = M_n$.

tower $\Rightarrow \forall\, m \leq n$, $E_m M_n = M_m$

$m, n \in \mathbb{N}$

$s, t \in [0, \infty)$.

$s \quad t \quad s$

Proposition 5.12. *Brownian motion is a martingale.*

Proof. $W \longrightarrow$ B.M.

WTS for every $s \leq t$, $E_s W_t = W_s$

Note $E_s W_t = E_s(W_t - W_s + W_s)$

$= E_s(W_t - W_s) + E_s W_s$

$= E(W_t - W_s) + W_s$

($\because$ W_s is $\mathcal{F}_s$ meas &

$W_t - W_s$ is ind of $\mathcal{F}_s$)

$$= 0 + W_s \qquad (\because W_t - W_s \sim N(0, t-s))$$

$$= W_s$$

QED.

6. Stochastic Integration

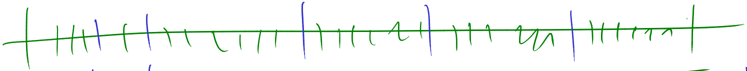
(Notation: Sometimes write $b_t = b(t)$)

6.1. Motivation.

at time t.

change in Stock price

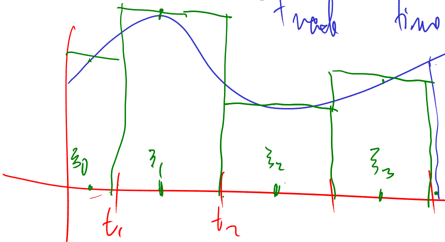
- Hold b_t shares of a stock with price S_t.
- Only trade at times $P = \{0 = t_1 < \dots, t_n = T\}$
- Net gain/loss from changes in stock price: $\displaystyle\sum_{k=0}^{n-1} b_{t_k} \Delta_k S$, where $\Delta_k S = S_{t_{k+1}} - S_{t_k}$.
- Trade continuously in time. Expect net gain/loss to be $\displaystyle\lim_{\|P\|\to 0} \sum_{k=0}^{n-1} b_{t_k} \Delta_k S = \int_0^T b_t \, dS_t$.
  - ▷ $\|P\| = \max_k (t_{k+1} - t_k)$. (Norm (P) $\approx$ mesh size(P))
  - ▷ Riemann-Stieltjes integral: $\displaystyle\lim_{\|P\|\to 0} \sum_{k=0}^{n-1} b_{\xi_k} \Delta_k S = \int_0^T b_t \, dS_t$,
  - ▷ The $\xi_k \in [t_k, t_{k+1}]$ can be chosen arbitrarily.
  - ▷ Only works if the *first variation* of S is finite. False for most stochastic processes.



$t_0 = 0 \quad t_1 \quad t_2 \quad \cdots \quad t_{n-1} \quad T = t_n$

trade times

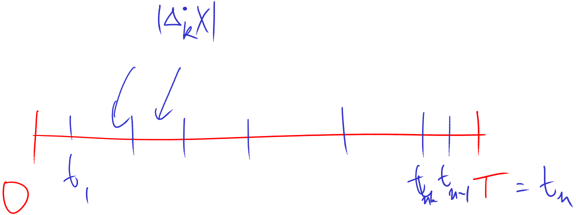
Some fn f.



$$\lim_{\|P\| \to 0} \sum f(\xi_k)(t_{k+1} - t_k)$$

$$(\text{Riemann int}) = \int_0^T f(t)\, dt$$

Riemann - Stieltjes int : $\underbrace{\lim_{\|P\|\to 0} \sum f(\xi_k)\left(S_{t_{k+1}} - S_{t_k}\right)}_{\int_0^T f(t)\, dS_t}$



$\Delta_k X = X_{t_{k+1}} - X_{t_k}$ = inc of X over $[t_k, t_{k+1}]$

6.2. **First Variation.**

Definition 6.1. For any process X, define the *first variation* by

$$V_{[0,T]}(X) \stackrel{\text{def}}{=} \lim_{\|P\|\to 0} \sum_{k=0}^{n-1} |\Delta_k X| \, . \stackrel{\text{def}}{=} \lim_{\|P\|\to 0} \sum_{k=0}^{n-1} |X_{t_{k+1}} - X_{t_k}| \, .$$

Remark 6.2. If $X(t)$ is a differentiable function of t then $V_{[0,T]}X < \infty$.

Proposition 6.3. $\boldsymbol{E} V_{[0,T]} W = \infty$

Remark 6.4. In fact, $V_{[0,T]}W = \infty$ almost surely. Brownian motion does not have finite first variation.

Remark 6.5. The Riemann-Stieltjes integral $\int_0^T b_t \, dW_t$ *does not exist.*

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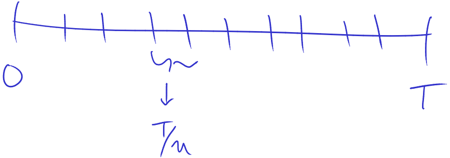
Let's check Prop 6.3:

N some large #.



Let $t_k = \frac{k}{n}$

$$E\, V_{[0,T]} W = E \lim_{n\to\infty} \sum \left| W_{t_{k+1}} - W_{t_k} \right| = \cancel{E} \lim_{N\to\infty} \sum \underbrace{E\left| W_{\frac{k+1}{n}} - W_{\frac{k}{n}} \right|}$$

Know $W_{\frac{k+1}{n}} - W_{\frac{k}{n}} \sim N\left(0, \frac{1}{n}\right)$

$$\Rightarrow E\left|W_{\frac{k+1}{n}} - W_{\frac{k}{n}}\right| = \underset{\substack{\uparrow \\ \text{some constant}}}{c} \cdot \left(\frac{1}{\sqrt{n}}\right)$$

$$\Rightarrow E\sum_{k=0}^{n-1}\left|W_{\frac{k+1}{n}} - W_{\frac{k}{n}}\right| = c\sum_{k=0}^{n-1}\frac{1}{\sqrt{n}} = c\sqrt{n} \xrightarrow{n\to\infty} \infty$$

(Note: If $X \sim N(0, \sigma^2)$,

then $$E|X| = \int_{-\infty}^{\infty} |x|\, e^{-x^2/2\sigma^2} \frac{dx}{\sqrt{2\pi}\,\sigma}$$

Put $y = \frac{x}{\sigma}$

$dx = \sigma\, dy$

$$= \int_{-\infty}^{\infty} |y| \sigma \, e^{-y^2/2} \frac{\cancel{\sigma}\, dy}{\sqrt{2\pi}\, \cancel{\sigma}}$$

$$= \sigma \underbrace{\int_{-\infty}^{\infty} |y| e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}}_{c}$$

$$\Rightarrow E|N(0,\sigma^2)| = \sigma \, (c)$$

6.3. Quadratic Variation.

Definition 6.6. If M is a continuous time adapted process, define

$$[M, M]_T = \lim_{\|P\| \to 0} \sum_{k=0}^{n-1} (M_{t_{k+1}} - M_{t_k})^2 = \lim_{\|P\| \to 0} \sum_{k=0}^{n-1} (\Delta_k M)^2 .$$

Proposition 6.7. *For continuous processes the following hold:*

1. *Finite first variation implies the quadratic variation is* 0
2. *Finite (non-zero) quadratic variation implies the first variation is infinite.*

(Will revisit this shortly)

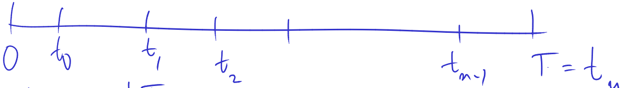
$[M, M]_T$ = Q.V. of M up to T

M adapted $\Rightarrow$ $[M, M]$ is an adapted (inc) process.

Proposition 6.8. $[W,W]_T = T$ *almost surely.*

Remark 6.9. For use in the proof: $\mathrm{Var}(\mathcal{N}(0,\sigma^2)^2) = \underbrace{\boldsymbol{E}\mathcal{N}(0,\sigma^2)^4}_{3\sigma^4} - (\boldsymbol{E}\underbrace{\mathcal{N}(0,\sigma^2)^2}_{\sigma^4})^2 = 2\sigma^4.$

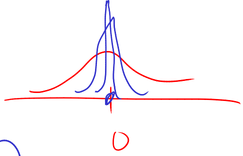
Proof:.



Set $t_k = \frac{kT}{n}$

QV: Want $\lim_{N \to \infty} \left[\sum_{0}^{n-1} (\Delta_k W)^2 - T \right] = 0$

(Will imply $[W,W]_T = T$).



Will show ① $E\left[\sum_0^{n-1}(\Delta_k W)^2 - T\right] = 0$

& ② $Var\left(\sum_0^{n-1}(\Delta_k W)^2 - T\right) \xrightarrow{N\to\infty} 0$

①+② $\Rightarrow \lim_{N\to\infty} \sum (\Delta_k W)^2 = T$ (i.e $[W,W]_T = T$).

Pf of ①: $E\left(\sum_0^{n-1}(\Delta_k W)^2 - T\right) =$

$$= \sum_{0}^{n-1} \frac{T}{n} - T = 0 \qquad \Delta_k W \sim N\left(0, \frac{T}{n}\right)$$

Pf of (2): $\mathrm{Var}\left(\sum_{0}^{n-1} (\Delta_k W)^2 - T\right) = \mathrm{Var}\left(\sum_{0}^{n-1} (\Delta_k W)^2\right)$

$$= \sum_{0}^{n-1} \mathrm{Var}\left[(\Delta_k W)^2\right] \qquad \left((\Delta_k W)^2 \sim N\left(0, \frac{T}{n}\right)^2\right.$$

$$= \sum_{0}^{n-1} 2 \frac{T^2}{n^2} = \frac{2T^2}{n} \xrightarrow{n \to \infty} 0 \qquad \left.\Rightarrow \mathrm{Var}\left((\Delta_k W)^2\right) = \underline{\underline{2}} \frac{T^2}{n^2}\right)$$

QED.

Proposition 6.10. $\underbrace{W_t^2}_{} - \underbrace{[W, W]_t}_{}$ *is a martingale.*

Pf: Know $[W, W]_t = t$.

$$\Rightarrow W_t^2 - [W, W]_t = W_t^2 - t$$

Let $M_t = W_t^2 - t$. NTS M is a mg

i.e. NTS $E_s(M_t) = M_s$

i.e. NTS $E_s(W_t^2 - t) = W_s^2 - s$

Pf: $E_s\left((W_t - W_s + W_s)^2 - t\right)$

$$= E_s\left(\underline{(W_t - W_s)^2} + W_s^2 + 2W_s(W_t - W_s)\right) - t$$

$$= E(W_t - W_s)^2 + W_s^2 + E_s\left[W_s(W_t - W_s)\right] - t$$

$$= t - s + W_s^2 + W_s\, \underbrace{E(W_t - W_s)}_{0} - t$$

(W_s is $\mathcal{F}_s$ meas)

($\because$ $\underline{W_t - W_s}$ is ind of $\mathcal{F}_s$)

$$= t - s + W_s^2 - t = W_s^2 - s$$

//

Theorem 6.11. *Let M be a continuous martingale.*

(1) $\boldsymbol{E} M_t^2 < \infty$ *if and only if* $\boldsymbol{E}[M, M]_t < \infty$.

(2) *In this case* $M_t^2 - [M, M]$ *is a continuous martingale.*

(3) *Conversely, if* $M_t^2 - A_t$ *is a martingale for any continuous, increasing process* A *such that* $A_0 = 0$, *then we must have* $A_t = [M, M]_t$.

Remark 6.12. The optional problem on HW2 gives some intuition in discrete time.