Def. X is a martingale, if

(1) X_n is adapted.

(2) $\mathbb{E}_n X_{n+1} = X_n$.

Ex. $\xi_1, \cdots, \xi_N$ i.i.d. $\mathbb{E}\xi_i = 0$.

$X_n = \xi_1 + \cdots + \xi_n$. — adapted

$\mathbb{E}_n X_{n+1} = \mathbb{E}_n(X_n + \xi_{n+1}) = \mathbb{E}_n X_n + \mathbb{E}_n \xi_{n+1} = X_n + \mathbb{E}\xi_{n+1} = X_n$ ✓

Ex. $X_1, \cdots, X_N$ independent, $X_i \geq 0$, $\mathbb{E}X_i = 1$.

$M_0 = 1$, $M_n = X_1 \cdot \cdots \cdot X_n$ — adapted

$\mathbb{E}_n M_{n+1} = \mathbb{E}_n(M_n \cdot X_{n+1}) = M_n \mathbb{E}_n X_{n+1} = M_n \mathbb{E} X_{n+1} = M_n$. ✓

Ex. X is a r.v. $X_n = \mathbb{E}_n X$. — adapted.

$\mathbb{E}_n X_{n+1} = \mathbb{E}_n(\mathbb{E}_{n+1} X) = \mathbb{E}_n X = X_n$. ✓

Thm 4.57 X_n is the wealth of a portfolio at $t=n$. The portfolio is self-financing iff. $D_n X_n$ is a MRT under $\widetilde{\mathbb{P}}$.

Recall: self-financing, if $\exists$ adapted Δ_n s.t. $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$

pf. ($\Rightarrow$)

$D_{n+1} = (1+r)^{-(n+1)}$

$$\widetilde{\mathbb{E}}_n(D_{n+1}X_{n+1}) = \widetilde{\mathbb{E}}_n\Big(D_{n+1}\big(\Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)\big)\Big)$$
$$= \widetilde{\mathbb{E}}_n(\Delta_n D_{n+1} S_{n+1}) + \widetilde{\mathbb{E}}_n\big(D_n(X_n - \Delta_n S_n)\big)$$
$$= \Delta_n \widetilde{\mathbb{E}}_n(D_{n+1} S_{n+1}) + D_n X_n - \Delta_n D_n S_n$$
$$= \cancel{\Delta_n D_n S_n} + D_n X_n - \cancel{\Delta_n D_n S_n}$$

($\Leftarrow$)

Define $\Delta_n = \dfrac{X_{n+1} - (1+r)X_n}{S_{n+1} - (1+r)S_n}$. Δ_n is $\mathcal{F}_{n+1}$ - measurable.

Goal: show Δ_n is $\mathcal{F}_n$ -measurable.

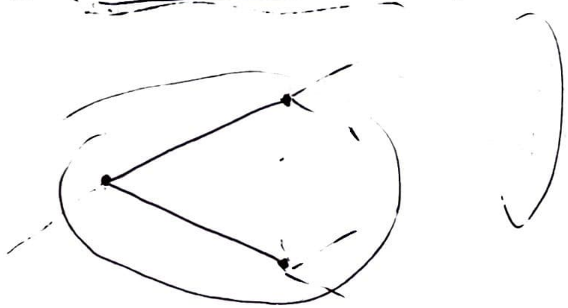
$$\Delta_n(S_{n+1} - (1+r)S_n) = X_{n+1} - (1+r)X_n.$$

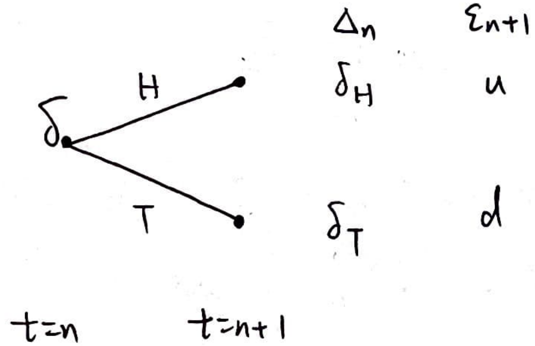
$$\widetilde{\mathbb{E}}_n\big(\Delta_n(S_{n+1} - (1+r)S_n)\big) = \widetilde{\mathbb{E}}_n\big(X_{n+1} - (1+r)X_n\big) = \widetilde{\mathbb{E}}_n X_{n+1} - (1+r)\cancel{\widetilde{\mathbb{E}}_n} X_n$$
$$= (1+r)^{n+1}\, \widetilde{\mathbb{E}}_n(D_{n+1}X_{n+1}) - (1+r)X_n = \underbrace{(1+r)^{n+1} D_n}_{=1+r} X_n - (1+r)X_n = 0.$$

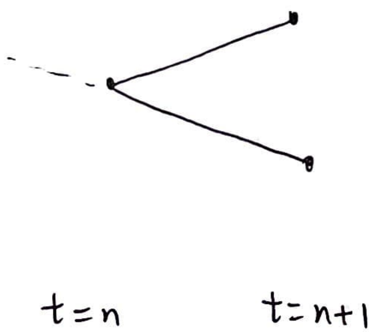
Let $S_{n+1} = S_n \varepsilon_{n+1}$, $\varepsilon_{n+1} = \begin{cases} u, & \text{if } \omega_{n+1} = H \\ d, & \text{if } \omega_{n+1} = T \end{cases}$.

$$0 = \mathbb{E}_n\Big(\Delta_n\big(S_{n+1} - (1+r)S_n\big)\Big) = \mathbb{E}_n\Big(\Delta_n\big(S_n\varepsilon_{n+1} - (1+r)S_n\big)\Big) = S_n\,\mathbb{E}_n\Big(\Delta_n\big(\varepsilon_{n+1} - (1+r)\big)\Big).$$

$$\Rightarrow \mathbb{E}_n\Big(\Delta_n\big(\varepsilon_{n+1} - (1+r)\big)\Big) = 0.$$







$$0 = \tilde{p}\cdot\delta_H\big(u-(1+r)\big) + \tilde{q}\cdot\delta_T\big(d-(1+r)\big)$$

$$\Rightarrow \tilde{p}\,\delta_H\,\frac{\big(u-(1+r)\big)}{u-d} = \tilde{q}\,\delta_T\,\frac{\big((1+r)-d\big)}{u-d}$$

$$\Rightarrow \tilde{p}\,\delta_H\,\tilde{q} = \tilde{q}\,\delta_T\,\tilde{p} \Rightarrow \delta_H = \delta_T =: \delta$$

$\Rightarrow \Delta_n$ is $\mathcal{F}_n$-measurable. $\boxtimes$

• Intuition of Lebesgue Integral.

Discrete case: $\mathbb{E}X = \sum_{w \in \Omega} X(w)\, p(w) = \sum_{x \in \mathrm{Range}(X)} x\, \mathbb{P}(X=x)$

General case: $\mathbb{E}X = \int_{\Omega} X(w)\, d\mathbb{P}(w) \underset{\text{if } X \text{ has pdf } p}{=\!=\!=} \int_{\mathbb{R}} x\, p(x)\, dx$

Let Δx be sufficiently small.

$$\mathbb{R} = \bigcup_{k=-\infty}^{\infty} (k\Delta x,\ (k+1)\Delta x]$$

$$\Omega_k = \{ k\Delta x < X \le (k+1)\Delta x \} = \{ w \in \Omega \mid k\Delta x < X(w) \le (k+1)\Delta x \}. \quad k \in \mathbb{Z}$$

$$\int_{\Omega} X(w)\, d\mathbb{P}(w) = \sum_{k \in \mathbb{Z}} \int_{\Omega_k} X(w)\, d\mathbb{P}(w) \approx \sum_{k \in \mathbb{Z}} \int_{\Omega_k} k\Delta x\, d\mathbb{P}(w) = \sum_{k \in \mathbb{Z}} k\Delta x \cdot \mathbb{P}(\Omega_k)$$

$$= \sum_{k \in \mathbb{Z}} k\Delta x\, \mathbb{P}\big(k\Delta x < X \le (k+1)\Delta x \big) = \sum_{k \in \mathbb{Z}} k\Delta x \int_{k\Delta x}^{(k+1)\Delta x} p(x)\, dx$$

$$\approx \sum_{k \in \mathbb{Z}} \int_{k\Delta x}^{(k+1)\Delta x} x\, p(x)\, dx = \int_{\mathbb{R}} x\, p(x)\, dx.$$

- Independence lemma. (Discrete)

X is independent of $\mathcal{F}_n$, Y is $\mathcal{F}_n$-measurable, $f: \mathbb{R}^2 \to \mathbb{R}$

$$\mathbb{E}_n f(X, Y) = \sum_{x \in \text{Range}} f(x, Y)\, \mathbb{P}(X = x) = g(Y)$$

$$g(y) = \mathbb{E} f(X, y) = \sum_{x \in \text{Range}} f(x, y)\, \mathbb{P}(X = x).$$

- Independence lemma. (Continuous)

X is independent of $\mathcal{F}_t$, Y is $\mathcal{F}_t$-measurable, $f: \mathbb{R}^2 \to \mathbb{R}$ "nice"

$\underline{\mathbb{E}_t f(X, Y)} = g(Y)$, $g(y) = \mathbb{E} f(X, y)$

If X has pdf $p_X(x)$, $g(y) = \int_{\mathbb{R}} f(x, y)\, p_X(x)\, dx$

$$\mathbb{E}_t f(X, Y) = \underline{\int_{\mathbb{R}} f(x, Y)\, p_X(x)\, dx}.$$

- Moment Generating Function (MGF).

X is r.v. $M_X : \mathbb{R} \to [0, \infty]$, $M_X(t) = \mathbb{E} e^{tX}$.

$X \sim N(0,1)$, pdf: $p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

$$M_X(t) = \int_{\mathbb{R}} e^{tx} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\underline{-x^2/2 + tx - t^2/2} + t^2/2}\, dx = \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{(x-t)^2}{2}}\, dx$$

$$(z = x - t) = \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{z^2}{2}}\, dz = e^{t^2/2}.$$

$X \sim N(0,1)$, $M_X(t) = e^{t^2/2}$.

$$\mathbb{E}\, e^{tX} = M_X(t)$$

$$M_X'(t) = \mathbb{E}(X e^{tX})$$

$$M_X''(t) = \mathbb{E}(X^2 e^{tX})$$

$$\vdots$$

$$M_X^{(n)}(t) = \mathbb{E}(X^n e^{tX})$$

$t=0$, $M_X^{(n)}(0) = \mathbb{E} X^n$.

Ex. $X, Y \sim N(0,1)$, independent. Compute $\mathbb{E}\, e^{\alpha XY}$, $\alpha \in \mathbb{R}$.

$\sigma(X) = \{ \{X \in A\} \mid A \subseteq \mathbb{R}, (A \text{ "nice"}) \}$

$$\mathbb{E}\, e^{\alpha XY} = \mathbb{E}\left(\mathbb{E}(e^{\alpha XY} \mid \sigma(X)) \right)$$

$\mathbb{E}(e^{\alpha XY} \mid \sigma(X)) = g(X)$, $g(x) = \mathbb{E}\, e^{\alpha x Y} = M_Y(\alpha x) = e^{(\alpha x)^2/2}$

$$\mathbb{E}\, e^{\alpha XY} = \mathbb{E}\, g(X) = \mathbb{E}\, e^{\frac{1}{2}\alpha^2 X^2} = \int_{\mathbb{R}} e^{\frac{1}{2}\alpha^2 x^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}(1-\alpha^2)x^2}\, dx.$$

1° $|\alpha| \geq 1$, $1-\alpha^2 \leq 0 \Rightarrow \mathbb{E}\, e^{\alpha XY} = \infty$.

2° $|\alpha| < 1$, $1-\alpha^2 > 0$, $z = \sqrt{1-\alpha^2}\, x$

$$\mathbb{E}\, e^{\alpha XY} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}z^2} \frac{dz}{\sqrt{1-\alpha^2}} = \frac{1}{\sqrt{1-\alpha^2}}.$$

Ex. Best Estimate in Mean Square Error.

X is r.v., for every $\mathcal{F}_t$-measurable r.v. Y,

$$\mathbb{E}(X - \mathbb{E}_t X)^2 \leq \mathbb{E}(X-Y)^2.$$

pf. $\mathbb{E}(X-Y)^2 = \mathbb{E}(X - \mathbb{E}_t X + \mathbb{E}_t X - Y)^2$

$$= \underbrace{\mathbb{E}(X - \mathbb{E}_t X)^2}_{\mathrm{I}} + \underbrace{2\,\mathbb{E}\big((X - \mathbb{E}_t X)(\mathbb{E}_t X - Y)\big)}_{\mathrm{II}} + \underbrace{\mathbb{E}(\mathbb{E}_t X - Y)^2}_{\mathrm{III}}.$$

$\mathrm{III} \geq 0$, $\mathrm{II} = \mathbb{E}\big((X - \mathbb{E}_t X)(\mathbb{E}_t X - Y)\big)$

$$= \mathbb{E}\Big(\mathbb{E}\big((X - \mathbb{E}_t X)\underline{(\mathbb{E}_t X - Y)} \,\big|\, \mathcal{F}_t\big)\Big)$$

$$\mathrm{II} = \mathbb{E}\Big((\mathbb{E}_t X - Y)\, \mathbb{E}(X - \mathbb{E}_t X \mid \mathcal{F}_t)\Big)$$
$$= \mathbb{E}\Big((\mathbb{E}_t X - Y)\, (\mathbb{E}_t X - \underline{\mathbb{E}_t(\mathbb{E}_t X)}) \Big)$$
$$= \mathbb{E}\Big((\mathbb{E}_t X - Y)\, (\mathbb{E}_t X - \mathbb{E}_t X) \Big) = 0.$$

$$\mathbb{E}(X-Y)^2 = \mathrm{I} + 2\,\mathrm{II} + \mathrm{III} = \mathrm{I} + \mathrm{III} \geq \mathrm{I} = \mathbb{E}(X - \mathbb{E}_t X)^2. \qquad \boxed{/\!/}$$

Ex. $X_1, \cdots, X_n$ i.i.d. r.v., $S_n = X_1 + \cdots + X_n$.

Find $\mathbb{E}(X_1 \mid S_n) = \mathbb{E}\big(X_1 \mid \sigma(S_n)\big) = g(S_n)$.

$$\underline{\mathbb{E}(X_i \mid S_n)} = \mathbb{E}(X_j \mid S_n), \quad 1 \leq i < j \leq n.$$

$$\mathbb{E}(X_1 \mid S_n) = \mathbb{E}(X_2 \mid S_n) = \cdots = \mathbb{E}(X_n \mid S_n)$$

$$n\,\mathbb{E}(X_1 \mid S_n) = \mathbb{E}(X_1 \mid S_n) + \cdots + \mathbb{E}(X_n \mid S_n)$$
$$= \mathbb{E}(X_1 + \cdots + X_n \mid S_n) = \mathbb{E}(S_n \mid S_n) = S_n$$

$$\Rightarrow \mathbb{E}(X_1 \mid S_n) = \frac{S_n}{n}.$$