Last time:

$$\tilde{P} \longrightarrow \tilde{E}_n\left(D_{n+1} S_{n+1}\right) = D_n S_n$$

(Discounted stock is a Mg under $\tilde{P}$)

$\tilde{P}$: Risk neutral Measure.

Theorem 4.57. *Let X_n represent the wealth of a portfolio at time n. The portfolio is self-financing portfolio if and only if the discounted wealth $D_n X_n$ is a martingale under the risk neutral measure $\tilde{\boldsymbol{P}}$.*

Remark 4.58. Recall a portfolio is *self financing* if $X_{n+1} = \boxed{\Delta_n} S_{n+1} + (1+r)(X_n - \Delta_n S_n)$ for some *adapted* process Δ_n.

(1) That is, self-financing portfolios use only tradable assets when trading, and don't look into the future.
(2) All replication has to be done using self-financing portfolios.

Last time: $X_n \rightarrow$ wealth if a self fin Port

$\Rightarrow D_n X_n$ is a mg under $\tilde{P}$.

Conversely: $D_n X_n$ mg under $\tilde{P} \Rightarrow X_n$ is self fin $\longrightarrow$ You try (No time)

Proof of Proposition 4.1. $\longrightarrow$ Security pays V_N at time N

Then AFP at time $n \leq N$ is

$$V_n = \frac{1}{D_n} \tilde{E}_n \left(D_N V_N \right).$$

Pf: Price by replication.

Find a self fin port $\left.\begin{array}{l} \text{Want } X_N = V_N. \\ \text{wealth at time } n \to X_n \end{array}\right\}$

Then we know $X_n = AFP$.

① Choose $X_N = V_N$

② Define $\underset{=}{X_n} = \underbrace{\frac{1}{D_n}\tilde{E}_n\left(D_N X_N\right)} = \frac{1}{\underline{D_n}}\tilde{E}_n\left(D_N \underline{V_N}\right)$

③ $\underline{\text{Claim}}$: $\underline{D_n X_n}$ is a Mg under $\underline{\tilde{P}}$

Pf: $\tilde{E}_n\left(\underline{D_{n+1}} X_{n+1}\right)$ $\underline{\text{Want}}$ $\underline{D_n X_n}$.

Know $\tilde{E}_n(D_{n+1} X_{n+1}) = \tilde{E}_n\left(\tilde{E}_{n+1}(D_N X_N)\right)$

$$\overset{\text{tower}}{=} \tilde{E}_n(D_N X_N) = \underline{D_n X_n}$$

Done !!

(4) Thm 4.5.7 $\Rightarrow$ X_n = wealth of a self fin Port.

Know $X_N = V_N$ $\Rightarrow$ Replication $\Rightarrow$ $\forall n \le N$, X_n = AFP.

Done !!

Example 4.59. Consider two stocks S^1 and S^2, $u = 2$, $d = 1/2$.

- ▷ The coin flips for S^1 are heads with probability 90%, and tails with probability 10%.
- ▷ The coin flips for S^2 are heads with probability 99%, and tails with probability 1%.
- ▷ Which stock do you like more?
- ▷ Amongst a call option for the two stocks with strike K and maturity N, which one will be priced higher?

Same!

Formula for $\tilde{p} = \dfrac{1+r-d}{u-d}$ ← doesnt depend on p, q !!

Remark 4.60. Even though the stock price changes according to a coin that flips heads with probability p_1, the arbitrage free price is computed using conditional expectations using the *risk neutral probability*. So when computing $\tilde{\boldsymbol{E}}_n V_N$, we use our new invented "risk neutral" coin that flips heads with probability $\tilde{p}_1$ and tails with probability $\tilde{q}_1$.

Concepts that will be generalized to continuous time.

- Probability measure: Lebesgue integral, and not a finite sum. Same properties.
- Filtration: Same intuition. No easy description.
- Conditional expectation: Same properties, no formula.
- Risk neutral measure: Formula for $\tilde{\boldsymbol{P}}$ is complicated (Girsanov theorem.)
- Everything still works because of of Theorem 4.57. Understanding why is harder.

Better way: $E\xi_i^2 = 1$

not essential.

5. Stochastic Processes

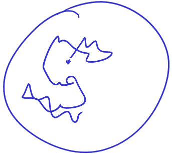
5.1. Brownian motion.

- Discrete time: Simple Random Walk.
  - ▷ $X_n = \sum_1^n \xi_i$, where ξ_i's are i.i.d. $\boldsymbol{E}\xi_i = 0$, and Range$(\xi_i) = \{\pm 1\}$.
- Continuous time: Brownian motion.
  - ▷ $Y_t = X_n + (t - n)\xi_{n+1}$ if $t \in [n, n+1)$.
  - ▷ Rescale: $Y_t^\varepsilon = \sqrt{\varepsilon} Y_{t/\varepsilon}$. (Chose $\sqrt{\varepsilon}$ factor to ensure $\mathrm{Var}(Y_t^\varepsilon) \approx t$.)
  - ▷ Let $W_t = \lim_{\varepsilon \to 0} Y_t^\varepsilon$.

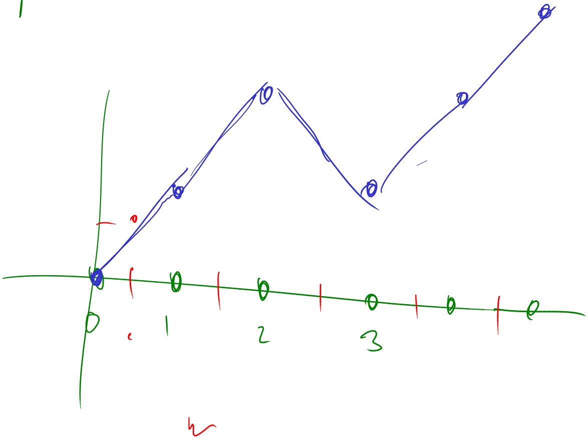
Definition 5.1 (Brownian motion)**.** The process W above is called a Brownian motion.

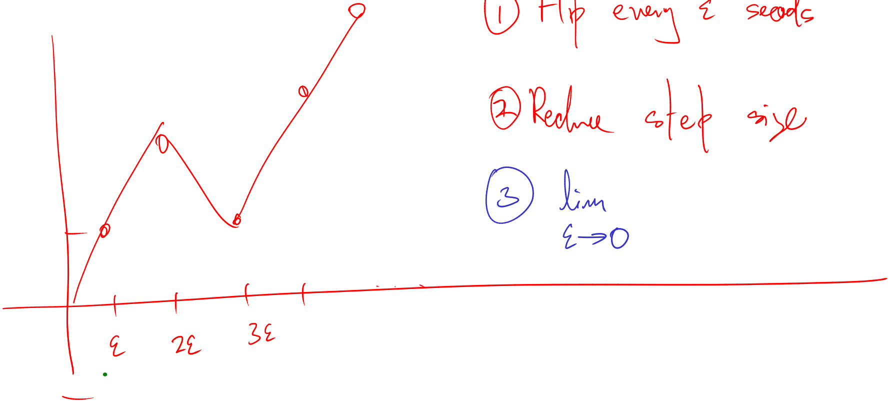
▷ Named after Robert Brown (a botanist).

▷ Definition is intuitive, but not as convenient to work with.



Simple random Walk





① Flip every ε seconds

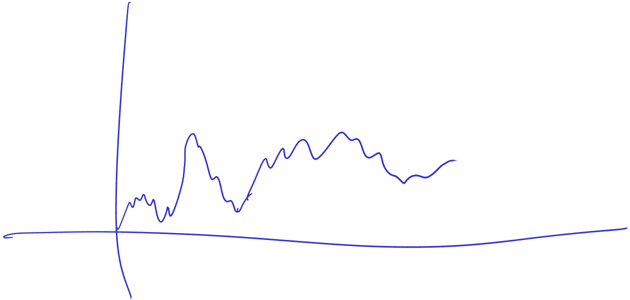
② Reduce step size

③ $\lim_{\varepsilon \to 0}$

$\varepsilon \to 0$ converge to a "cts time RW"

(Brownian motion)

u



$Y_{t/\varepsilon}$ (say $t/\varepsilon \in \mathbb{N}$)

$$Y_{t/\varepsilon} = \sum_{1}^{t/\varepsilon} \xi_i$$

(sum of $\frac{t}{\varepsilon}$ iid RV's mean 0 & Var 1)

$$\operatorname{Var}(Y_{t/\varepsilon}) = \frac{t}{\varepsilon}$$

$$\Rightarrow \operatorname{Var}(\sqrt{\varepsilon}\, Y_{t/\varepsilon}) = (\sqrt{\varepsilon})^2 \cdot \frac{t}{\varepsilon} = t$$

- If t, s are multiples of ε: $Y_t^\varepsilon - Y_s^\varepsilon \sim \sqrt{\varepsilon} \underbrace{\sum_{i=1}^{(t-s)/\varepsilon} \xi_i}_{} \xrightarrow{\varepsilon \to 0} \mathcal{N}(0, t-s)$. (CLT)
- $Y_t^\varepsilon - Y_s^\varepsilon$ only uses coin tosses that are "after s", and so independent of Y_s^ε.

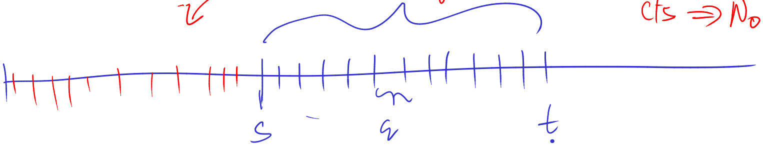
Definition 5.2. Brownian motion is a *continuous process* such that:

(1) $W_t - W_s \sim \mathcal{N}(0, t-s)$,

(2) $W_t - W_s$ is independent of $\mathcal{F}_s$. ← Inf from the times before s.

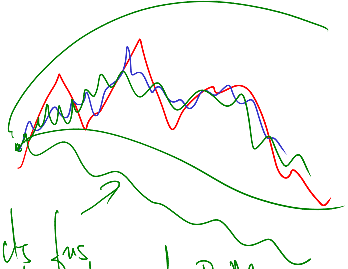
Ind of

cts ⇒ No Jumps



5.2. Sample space, measure, and filtration.

- Discrete time: Sample space $\Omega = (\omega_1, \dots, \omega_N)$. ω_i = outcome of ith coin toss
- View $(\omega_1, \dots, \omega_N)$ as the trajectory of a random walk.
- Continuous time: Sample space $\Omega = C([0, \infty))$ (space of continuous functions).
  - ▷ It's infinite. No probability mass function!
  - ▷ Mathematically impossible to define $\boldsymbol{P}(A)$ for *all* $A \subseteq \Omega$.



View cts fns as trajectories of BM.

SRW corresponding to ω.

- Restrict our attention to $\mathcal{G}$, a subset of some sets $A \subseteq \Omega$, on which $\boldsymbol{P}$ can be defined.
  - ▷ $\mathcal{G}$ is a σ-algebra. (Closed countable under unions, complements, intersections.)
- $\boldsymbol{P}$ is called a *probability measure* on $(\Omega, \mathcal{G})$ if:
  - ▷ $\boldsymbol{P} : \mathcal{G} \to [0,1]$, $\boldsymbol{P}(\emptyset) = 0$, $\boldsymbol{P}(\Omega) = 1$. (i.e. $\forall A \in \mathcal{G}$, $P(A) \in [0,1]$)
  - ▷ $\boldsymbol{P}(A \cup B) = \boldsymbol{P}(A) + \boldsymbol{P}(B)$ if $A, B \in \mathcal{G}$ are disjoint.
  - ▷ If $A_n \in \mathcal{G}$, $\boldsymbol{P}\left(\bigcup_1^\infty A_n\right) = \lim_{n\to\infty} \boldsymbol{P}(A_n)$.
- Random variables are *measurable* functions of the sample space:
  - ▷ Require $\{X \in A\} \in \mathcal{G}$ for every "nice" $A \subseteq \mathbb{R}$.
  - ▷ E.g. $\{X = 1\} \in \mathcal{G}$, $\{X > 5\} \in \mathcal{G}$, $\{X \in [3,4)\} \in \mathcal{G}$, etc.
  - ▷ Recall $\{X \in A\} = \{\omega \in \Omega \mid X(\omega) \in A\}$.

$$\{\omega \in \Omega \mid X(\omega) > 0\} = \{X > 0\}$$

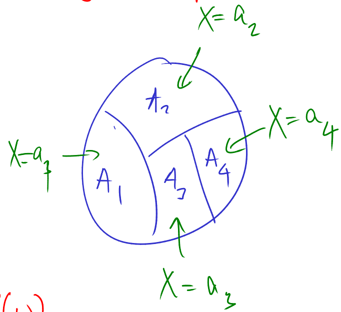
some subset of Ω.

- Expectation is a *Lebesgue Integral*: Notation $\boldsymbol{E}X = \int_\Omega X\, d\boldsymbol{P} = \int_\Omega X(\omega) d\boldsymbol{P}(\omega)$.
  - ▷ No simple formula.
  - ▷ If $X = \sum a_i \mathbf{1}_{A_i}$, then $\boldsymbol{E}X = \sum a_i \boldsymbol{P}(A_i)$. $(a_i \in \mathbb{R},\ A_i \in \mathcal{G}$ are disj$)$
  - ▷ $\mathbf{1}_A$ is the *indicator function* of A: $\mathbf{1}_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$

$EX = \sum a_i P(X = a_i)$

Discrete case: $EX = \sum_{\omega \in \Omega} X(\omega) P(\omega)$

$\omega \in \Omega$ $dP(\omega)$



almost surely: event of prob 1.

Proposition 5.3 (Useful properties of expectation).

(1) (Linearity) $\alpha, \beta \in \mathbb{R}$, X, Y *random variables*, $\boldsymbol{E}(\alpha X + \beta Y) = \alpha \boldsymbol{E} X + \beta \boldsymbol{E} Y$.

(2) (Positivity) *If* $X \geqslant 0$ *then* $\boldsymbol{E} X \geqslant 0$. *If* $X \geqslant 0$ *and* $\boldsymbol{E} X = 0$ *then* $X = 0$ *almost surely*.

$(P(X=0) = 1)$.

(3) (Layer Cake) *If* $X \geqslant 0$, $\boxed{\boldsymbol{E} X = \int_0^\infty \boldsymbol{P}(X \geqslant t)\, dt.}$

(4) *More generally, if* φ *is increasing,* $\varphi(0) = 0$ *then* $\boldsymbol{E}\varphi(X) = \int_0^\infty \varphi'(t)\, \boldsymbol{P}(X \geqslant t)\, dt.$

$(X \geqslant 0)$

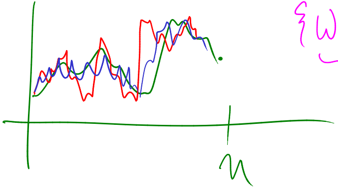
(5) (Unconscious Statistician Formula) *If PDF of* X *is* p, *then* $\boldsymbol{E} f(X) = \int_{-\infty}^{\infty} f(x) p(x)\, dx.$

(lazy)

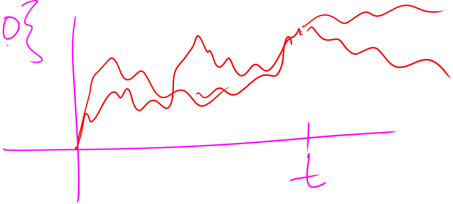
Know $EX = \int x\, p(x)\, dx$

$E f(X) = \int f(x)\, p(x)\, dx$

- Filtrations:
  - ▷ Discrete time: $\mathcal{F}_n$ = events described using the first n coin tosses.
  - ▷ Coin tosses doesn't translate well to continuous time.
  - ▷ Discrete time try #2: $\mathcal{F}_n$ = events described using the *trajectory* of the SRW up to time n.
  - ▷ Continuous time: $\boxed{\mathcal{F}_t}$ = events described using the *trajectory* of the *Brownian motion* up to time t.
  - ▷ If $t_i \leqslant t$, $A_i \subseteq \mathbb{R}$ then $\{W_{t_1} \in A_1, \dots, W_{t_n} \in A_n\} \in \mathcal{F}_t$. (Need all $t_i \leqslant t$!)
  - ▷ As before: if $s \leqslant t$, then $\mathcal{F}_s \subseteq \mathcal{F}_t$. ← Filteration.
  - ▷ Discrete time: $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Continuous time: $\mathcal{F}_0 = \{A \in \mathcal{G} \mid \boldsymbol{P}(A) \in \{0, 1\}\}$.



$\{W_{100} = 0\}$



fix $t \in \mathbb{R}$.

$A = [0, \infty)$

~~$A \subseteq \mathbb{R}$ nice~~

$\{ W_s \in \cancel{A} \}$ & $\underbrace{s \le t}$

$\{ W_s > 0 \} \in \mathcal{F}_t$? $\leftarrow$ Yes.

5.3. Conditional expectation.

- Notation $\boldsymbol{E}_t(X) = \boldsymbol{E}(X \mid \mathcal{F}_t)$ (read as conditional expectation of X given $\mathcal{F}_t$)
- No formula! But same intuition as discrete time.
- $\boldsymbol{E}_t X(\omega) =$ "average of X over $\Pi_t(\omega)$", where $\Pi_t(\omega) = \{\omega' \in \Omega \mid \omega'(s) = \omega(s) \ \forall s \leqslant t\}$.
- Mathematically problematic: $\boldsymbol{P}(\Pi_t(\omega)) = 0$ (but it still works out.)

Definition 5.4. $\boldsymbol{E}_t X$ is the unique *random variable* such that:

(1) $\boldsymbol{E}_t X$ is $\mathcal{F}_t$-measurable. (i.e $\forall A \subseteq \mathbb{R}$, $\{E_t X \in A\} \in \mathcal{F}_t$)

(2) For every $A \in \mathcal{F}_t$, $\displaystyle\int_A \boldsymbol{E}_t X \, d\boldsymbol{P} = \int_A X \, d\boldsymbol{P}$ (Disc time $\displaystyle\sum_{\omega\in A} E_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$)

Remark 5.5. Choosing $A = \Omega$ implies $\boldsymbol{E}(\boldsymbol{E}_t X) = \boldsymbol{E}X$.

Proposition 5.6 (Useful properties of conditional expectation)**.**

(1) If $\alpha, \beta \in \mathbb{R}$ are constants, X, Y, random variables $\boldsymbol{E}_t(\alpha X + \alpha Y) = \alpha \boldsymbol{E}_t X + \beta \boldsymbol{E}_t Y$.

(2) If $X \geqslant 0$, then $\boldsymbol{E}_t X \geqslant 0$. Equality holds if and only if $X = 0$ almost surely.

(3) (Tower property) *If $0 \leqslant s \leqslant t$, then $\boldsymbol{E}_s(\boldsymbol{E}_t X) = \boldsymbol{E}_s X$.*

(4) If X is $\mathcal{F}_t$ measurable, and Y is any random variable, then $\boldsymbol{E}_t(XY) = X\boldsymbol{E}_t Y$.

(5) If X is $\mathcal{F}_t$ measurable, then $\boldsymbol{E}_t X = X$ (follows by choosing $Y = 1$ above).

(6) If Y is independent *of $\mathcal{F}_t$, then $\boldsymbol{E}_t Y = \boldsymbol{E}Y$.*

Remark 5.7. These properties are exactly the same as in discrete time.