Last time: $\boxed{\text{No Arb}}$ (assumption)

Bond

Stocks

Market

NTA (option)

AFP: V_0 $\longrightarrow$ if when allowed to trade the NTA at price V_0 the extended market remains arb free.

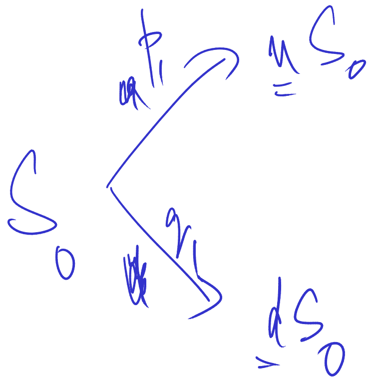
Replication!

If the payoff of the NTA can be replicated using only tradable assets

then X_{0n} = initial wealth of the rep port

$= V_{0n} =$ AFP.

Last time: Binom model: Multi period version



Formula: Payoff $= V_N$

$$\boxed{AFP = \frac{1}{D_n}\tilde{E}_n(D_N V_N)}$$

4.4. **Conditional expectation.**

Definition 4.28. Let X be a random variable, and $n \leqslant N$. We define $\boldsymbol{E}(X \mid \mathcal{F}_n) = \boldsymbol{E}_n X$ to be the *random variable* given by

$$\boldsymbol{E}_n X(\omega) = \sum_{x_i \in \mathrm{Range}(X)} x_i \boldsymbol{P}(X = x_i \mid \Pi_n(\omega))$$

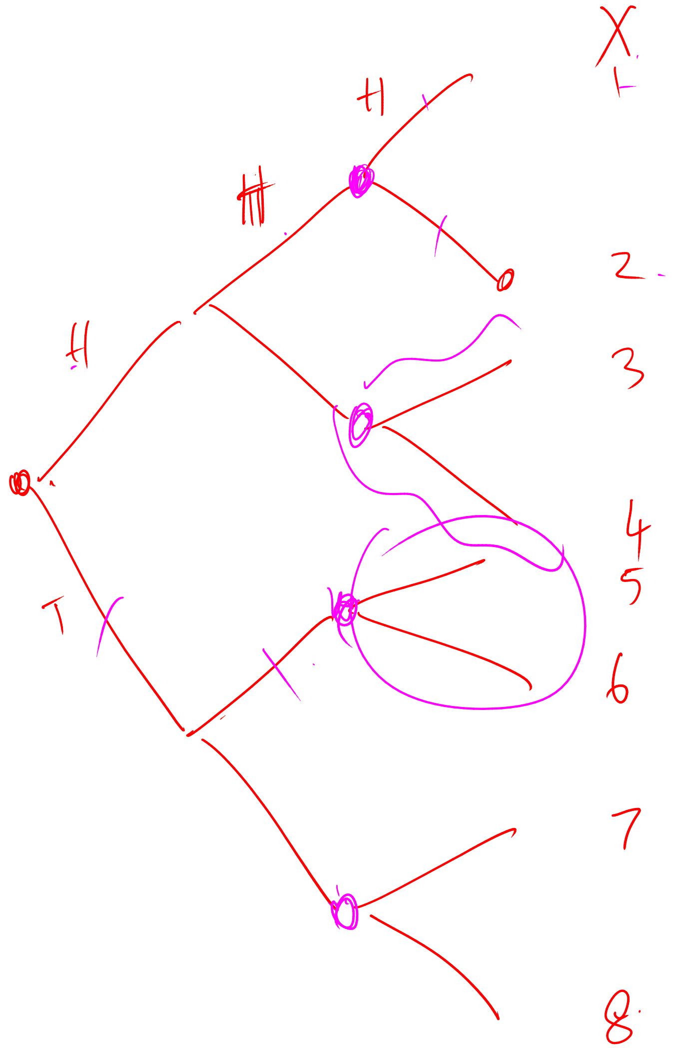
$$\text{where} \quad \Pi_n(\omega) = \{\omega' \in \Omega \mid \omega'_1 = \omega_1, \ \ldots, \omega'_n = \omega_n\}$$

Remark 4.29. The above formula does not generalize well to infinite probability spaces. We will develop certain properties of $\boldsymbol{E}_n$, and then only use those properties going forward.

Example 4.30. If we represent Ω as a tree, $\boldsymbol{E}_n X$ can be computed by averaging over leaves.

Remark 4.31. $\boldsymbol{E}_n X$ is the "best approximation" of X given only the first n coin tosses.

$E_n X \longrightarrow$ "Best approximation of X given info up to time n"



$\underline{E_2 X}$

$p_1 \cdot 1 + q_1 \cdot 2$

$p_1 \cdot 3 + q_1 \cdot 4$

$p_1\, 3 + q_1 \cdot 4$

(3 coin tosses)

Proposition 4.32. *The conditional expectation $\boldsymbol{E}_n X$ defined by the above formula satisfies the following two properties:*

(1) $\boldsymbol{E}_n X$ *is an $\mathcal{F}_n$-measurable* random variable.

(2) *For every $A \in \mathcal{F}_n$,* $\displaystyle\sum_{\omega\in A} \boldsymbol{E}_n X(\omega) p(\omega) = \sum_{\omega \in A} X(\omega) p(\omega)$.

($E_n X$ dep only on first n coin tosses.)

Remark 4.33. This property is used to define conditional expectations in the continuous time setting. It turns out that there is exactly one random variable that satisfies both the above properties; and thus we define $\boldsymbol{E}_n X$ to be the unique random variable which satisfies both the above properties.

Remark 4.34. Note, choosing $A = \Omega$, we see $\boxed{\boldsymbol{E}(\boldsymbol{E}_n X) = \boldsymbol{E} X}$.

On A: Avg of $X = \cancel{\frac{1}{P(A)}} \sum_{\omega \in A} X(\omega) p(\omega)$

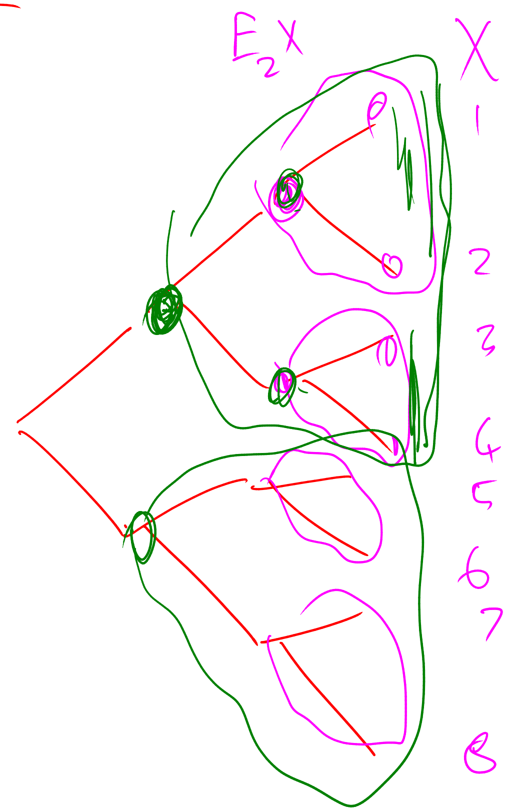
equal.

Avg of $E_n X$ (on A) $= \cancel{\frac{1}{P(A)}} \sum_{\omega \in A} E_n X(\omega) p(\omega)$

Proposition 4.35. *(1) If X, Y are two random variables and $\alpha \in \mathbb{R}$, then* $\boldsymbol{E}_n(X + \alpha Y) = \boldsymbol{E}_n X + \alpha \boldsymbol{E}_n Y$.

(2) (Tower property) *If $m \leqslant n$, then* $\boldsymbol{E}_m(\boldsymbol{E}_n X) = \boldsymbol{E}_m X$.

(3) If X is $\mathcal{F}_n$ measurable, and Y is any random variable, then $\boxed{\boldsymbol{E}_n(XY) = X \boldsymbol{E}_n Y}$.

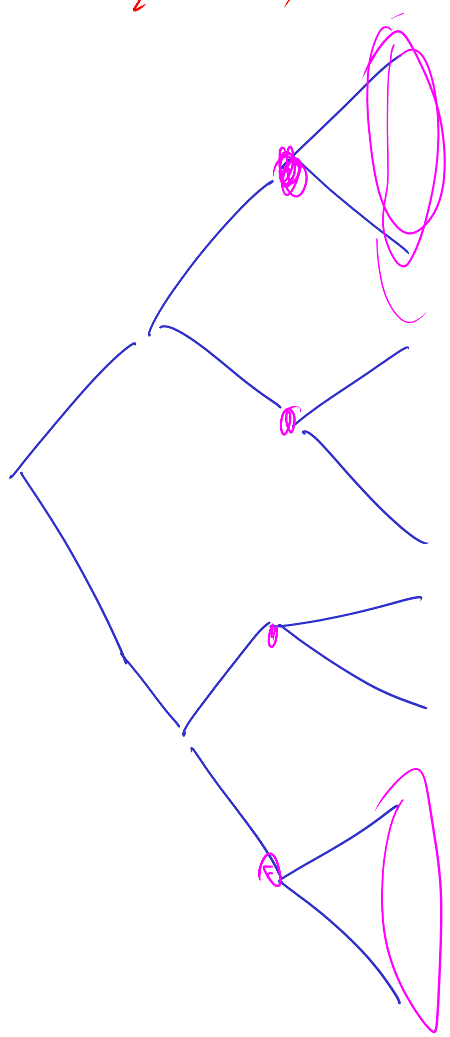


$E_2 X$

$E_1 X$

h

(X is $\mathcal{F}_2$ meas) $\longrightarrow$



X	Y	$E_2(XY)$
100	1	$100(1p_1 + 2q_1)$
100	2	
300	3	
300	4	
-10	5	
-10	6	
5	7	
5	8	

Proposition 4.36. *(1) If X is measurable with respect to $\mathcal{F}_n$, then $\boldsymbol{E}_n X = X$.*
(2) If X is independent of $\mathcal{F}_n$ then $\boxed{\boldsymbol{E}_n X = \boldsymbol{E} X}$.

Remark 4.37. We say X is independent of $\mathcal{F}_n$ if for every $A \in \mathcal{F}_n$ and $B \subseteq \mathbb{R}$, the events A and $\{X \in B\}$ are independent.

Example 4.38. If X only depends on the $(n+1)^{\text{th}}$, $(n+2)^{\text{th}}$, ..., N^{th} coin tosses and *not* the 1^{st}, 2^{nd}, ..., n^{th} coin tosses, then X is independent of $\mathcal{F}_n$.

Notation: $\{X \in B\} = \{\omega \mid X(\omega) \in B\}$.

Proposition 4.39 (Independence lemma)**.** *If X is independent of $\mathcal{F}_n$ and Y is $\mathcal{F}_n$-measurable, and $f : \mathbb{R}^2 \to \mathbb{R}$ is a function then*

$$\boldsymbol{E}_n f(X, Y) = \sum_{i=1}^{m} f(x_i, Y)\boldsymbol{P}(X = x_i)\,, \qquad \textit{where } \{x_1, \ldots, x_m\} = X(\Omega)\,.$$

Range (X)

Say $Y = y$ (some const)

$$E_n f(X, y) = \boxed{E f(X, y)} = \sum_{x_i \in \text{Range}(X)} f(x_i, y)\, P(X = x_i)$$

Y $\qquad$ Y $\qquad$ Y

Avege the ind RV treating the meas RV as a const.

4.5. **Martingales.**

Definition 4.40. A *stochastic process* is a collection of random variables $X_0, X_1, \ldots, X_N$.

Example 4.41*.* Typically X_n is the wealth of an investor at time n, or S_n is the price of a stock at time n.

Definition 4.42. A stochastic process is *adapted* if X_n is $\mathcal{F}_n$-measurable for all n. (Non-anticipating.)

Remark 4.43*.* Requiring processes to be adapted is fundamental to Finance. Intuitively, being adapted forbids you from trading today based on tomorrows stock price. All processes we consider (prices, wealth, trading strategies) will be adapted.

Example 4.44 (Money market)*.* Let $Y_0 = Y_0(\omega) = a \in \mathbb{R}$. Define $Y_{n+1} = (1+r)Y_n$. (Here r is the interest rate.)

Example 4.45 (Stock price)*.* Let $S_0 \in \mathbb{R}$. Define $S_{n+1}(\omega) = \begin{cases} uS_n(\omega) & \omega_{n+1} = 1\,, \\ dS_n(\omega) & \omega_{n+1} = -1\,. \end{cases}$

S_n is a stoch process

S_n is an ADAPTED process

$Y_n = (1+r)^n a.$

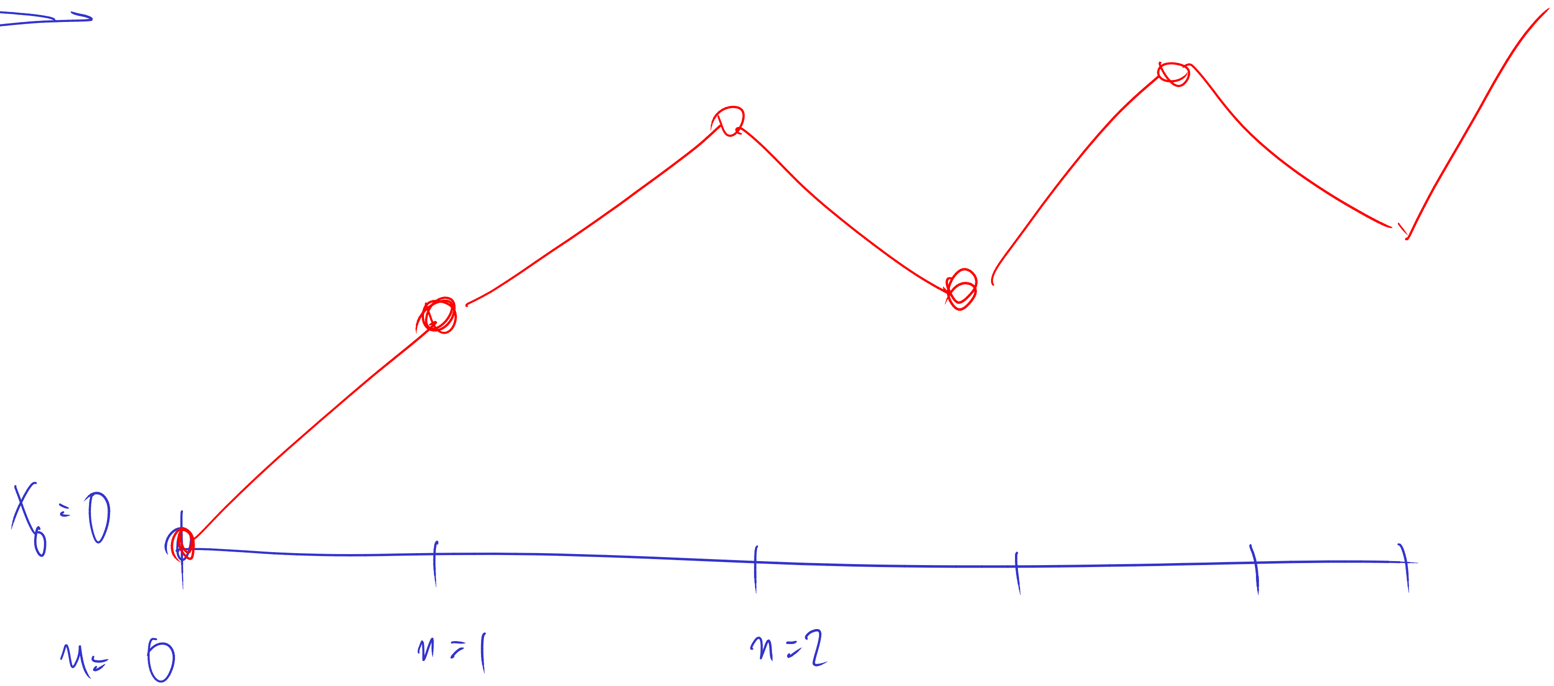
(adapted process)

Definition 4.46. We say an adapted process M_n is a martingale if $\boldsymbol{E}_n M_{n+1} = M_n$. (Recall $\boldsymbol{E}_n Y = \boldsymbol{E}(Y \mid \mathcal{F}_n)$.)

Remark 4.47. Intuition: A martingale is a "fair game".

$E\xi_n = 0 \ \forall n.$

Example 4.48 (Unbiased random walk). If $\xi_1, \dots, \xi_N$ are i.i.d. and mean zero, then $X_n = \sum_{k=1}^{n} \xi_k$ is a martingale.



Check X is a mg: Note $X_{n+1} = X_n + \xi_{n+1}$

$$\Rightarrow E_n(X_{n+1}) = E_n(X_n + \xi_{n+1})$$

$$= \underbrace{E_n X_n}_{X_n \text{ is } \mathcal{F}_n \text{ meas}} + \underbrace{E_n \xi_{n+1}}_{\xi_{n+1} \text{ is ind of } \mathcal{F}_n.}$$

$$= X_n + E_{\cancel{n}} \xi_{n+1}$$

0

$$\Rightarrow E_n X_{n+1} = X_n$$

$\Rightarrow X$ is a mg!

Remark 4.49. If M is a martingale, then for every $m \leqslant n$, we must have $\boldsymbol{E}_m M_n = M_m$.

Remark 4.50. If M is a martingale then $\boldsymbol{E} M_n = \boldsymbol{E} M_0 = M_0$.

$$E_n M_{n+2} \overset{\text{tower}}{=} E_n E_{n+1} M_{n+2} \overset{\text{Mg}}{=} E_n M_{n+1} \overset{\text{Mg}}{=} M_n$$

Note $\quad E M_n = E_0 M_n \overset{4.49}{=} M_0$

4.6. **Change of measure.**

- Gambling in a Casino: If it's a martingale, then on average you won't make or lose money.
- Stock market: Bank always pays interest! Not looking for a "break even" strategy.
- Mathematical tool that helps us price securities: Find a *Risk Neutral Measure.*
  - ▷ Discounted stock price is (usually) not a martingale.
  - ▷ Invent a "risk neutral measure" which the discounted stock price is a martingale.
  - ▷ Securities can be priced by taking a conditional expectation *with respect to the risk neutral measure.* (That's the meaning of $\tilde{\boldsymbol{E}}_n$ in Proposition 4.1.)

AFP at time $n = \frac{1}{D_n} \tilde{E}_n (D_N V_D)$

Cond exp wrt the "Risk Neutral Measure"

Definition 4.51. Let $D_n = (1+r)^{-n}$ be the discount factor. (So D_n\$ in the bank at time 0 becomes 1\$ in the bank at time n.)

- Invent a new probability mass function $\tilde{p}$.
- Use a tilde to distinguish between the new, invented, probability measure and the old one.
  - ▷ $\tilde{\boldsymbol{P}}$ the probability measure obtained from the PMF $\tilde{p}$ (i.e. $\tilde{\boldsymbol{P}}(A) = \sum_{\omega \in A} \tilde{p}(\omega)$).
  - ▷ $\tilde{\boldsymbol{E}}$, $\tilde{\boldsymbol{E}}_n$ conditional expectation with respect to $\tilde{\boldsymbol{P}}$ (the new "risk neutral" coin)

Definition 4.52. We say $\boldsymbol{P}$ and $\tilde{\boldsymbol{P}}$ are equivalent if for every $A \in \mathcal{F}_N$, $\boldsymbol{P}(A) = 0$ if and only if $\tilde{\boldsymbol{P}}(A) = 0$.

Definition 4.53. A *risk neutral measure* is an equivalent measure $\tilde{\boldsymbol{P}}$ under which $D_n S_n$ is a martingale. (I.e $\tilde{\boldsymbol{E}}_n(D_{n+1}S_{n+1}) = D_n S_n$. (for all n))

Remark 4.54. If there are more than one risky assets, S^1, ..., S^k, then we require $D_n S_n^1$, ..., $D_n S_n^k$ to all be martingales under the risk neutral measure $\tilde{\boldsymbol{P}}$.

Remark 4.55. Proposition 4.1 says that any security with payoff V_N at time N has arbitrage free price $V_n = \frac{1}{D_n} \tilde{\boldsymbol{E}}_n(D_N V_N)$ at time n. (Called the risk neutral pricing formula.)

tilde

Proposition 4.56. *Let $\tilde{\boldsymbol{P}}$ be an equivalent measure under which the coins are i.i.d. and land heads with probability $\tilde{p}_1$ and tails with probability $\tilde{q}_1 = 1 - \tilde{p}_1$.*

(1) *Under $\tilde{\boldsymbol{P}}$, we have $\tilde{\boldsymbol{E}}_n(D_{n+1}S_{n+1}) = \frac{\tilde{p}_1 u + \tilde{q}_1 d}{1+r} D_n S_n$.* $\circledast$

(2) *$\tilde{\boldsymbol{P}}$ is the risk neutral measure if and only if $\tilde{p}_1 u + \tilde{q}_1 d = 1 + r$. (Explicitly $\tilde{p}_1 = \frac{1+r-d}{u-d}$, and $\tilde{q}_1 = \frac{u-(1+r)}{u-d}$.)*

$$\textcircled{1} \quad \tilde{E}_n\left(D_{n+1} S_{n+1}\right) = (1+r)^{-(n+1)} \tilde{E}_n S_{n+1}$$

$$= (1+r)^{-(n+1)} \tilde{E}_n\left(S_n X_{n+1}\right)$$

where $X_{n+1} = \begin{cases} u & \omega_{n+1} = 1 \\ d & \omega_{n+1} = -1 \end{cases}$

$$= (1+r)^{-(n+1)} \tilde{E}_n\left(S_n X_{n+1}\right)$$

A

(S_n is $\mathcal{F}_n$ meas, X_{n+1} is ind)

$$= (1+r)^{-(n+1)} S_n \tilde{\mathbb{E}} X_{n+1}$$

$$= (1+r)^{-n+1} S_n \left(\tilde{p}_1 u + \tilde{r}_1 d \right)$$

$$= \left(\frac{\tilde{p}_1 u + \tilde{r}_1 d}{1+r} \right) D_n S_n$$

//

Theorem 4.57. *Let X_n represent the wealth of a portfolio at time n. The portfolio is self-financing portfolio if and only if the discounted wealth $D_n X_n$ is a martingale under the risk neutral measure $\tilde{\boldsymbol{P}}$.*

Remark 4.58. Recall a portfolio is *self financing* if $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$ for some *adapted* process Δ_n.

(1) That is, self-financing portfolios use only tradable assets when trading, and don't look into the future.

(2) All replication has to be done using self-financing portfolios.

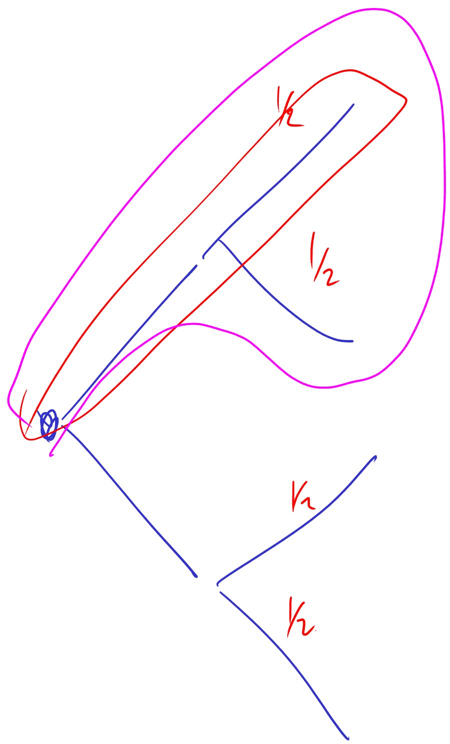
Check: Self fin $\Longrightarrow$ $D_n X_n$ is a $\tilde{P}$ mg.

$$\tilde{E}_n\left(D_{n+1} X_{n+1}\right) = \tilde{E}_n\left(D_{n+1}\left(\Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)\right)\right)$$

$$= \Delta_n \underbrace{\tilde{E}_n\left(D_{n+1} S_{n+1}\right)}_{D_n S_n} + D_n\left(X_n - \Delta_n S_n\right)$$

$$= D_n X_n$$

QED.



X	$E_Y X$
1	3/2
	3/2
2	
3	7/2
4	7/2

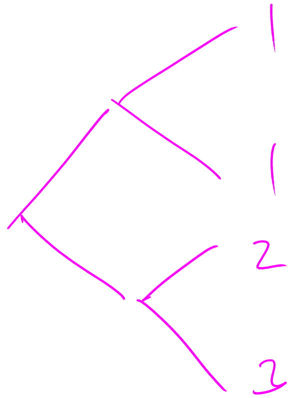
$A = \{(1,1)\} \in \mathcal{F}_2$

$\notin \mathcal{F}_1$

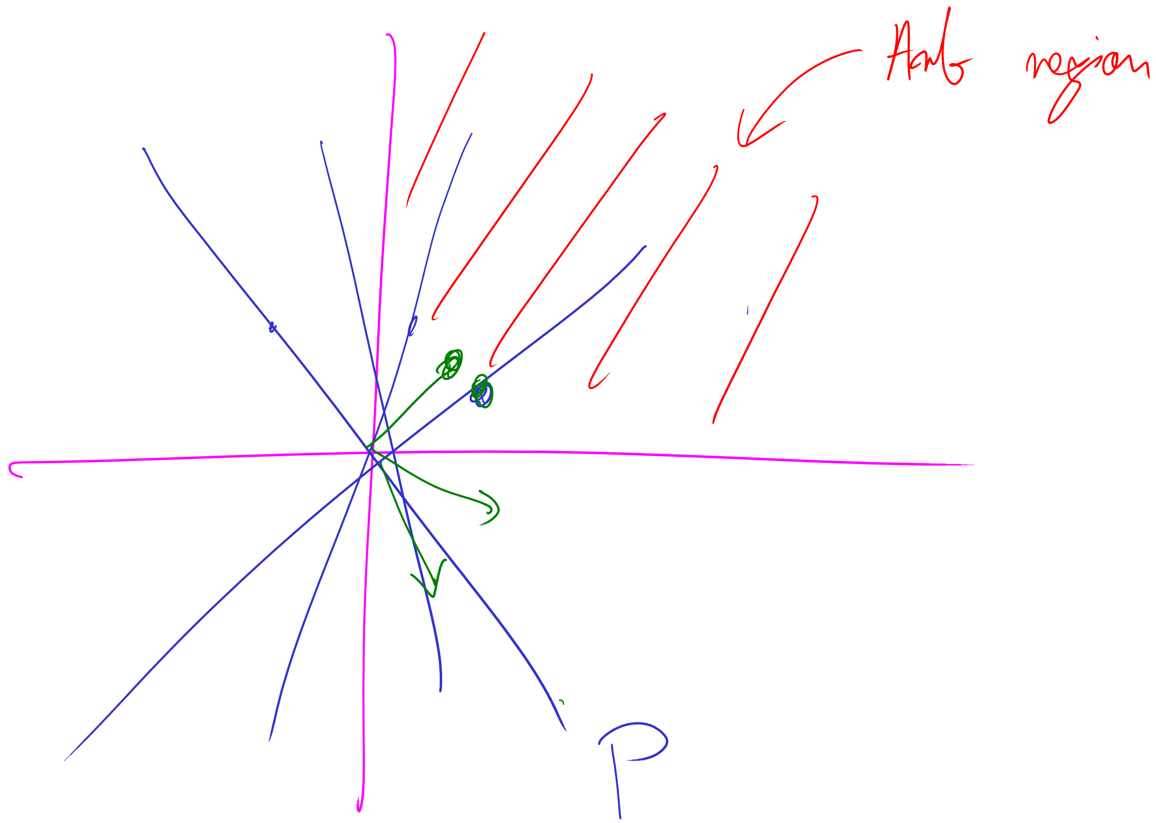
$B = \{(1,1),(1,-1)\} \in \mathcal{F}_1$

$$\sigma(X) = \{ \{X \in B\} \mid B \subseteq \mathbb{R} \}$$

X



$$\sigma(X) = \{ \{(1,1),(1,-1)\}, \{(-1,1)\}, \{(-1,-1)\}, \{(-1,1),(-1,-1)\}, \emptyset, \Omega \}$$



Anti region
P