~~Prop 3.5~~

Prop 3.3 $\quad V_1$ at $t=1 \quad V_1(H),\ V_1(T)$

$$V_0 = \frac{1}{1+r}\left(\tilde{p}_1 V_1(H) + \tilde{q}_1 V_1(T)\right) = \frac{1}{1+r}\tilde{\mathbb{E}} V_1$$

$$\tilde{p}_1 = \frac{(1+r)-d}{u-d}, \quad \tilde{q}_1 = \frac{u-(1+r)}{u-d}. \quad \Delta_0 = \frac{V_1(H)-V_1(T)}{(u-d)S_0}.$$

$$\begin{tikzcd}[row sep=small] & \text{Stock: } \Delta_0 \text{ shares} \\ X_0 \arrow[ur] \arrow[dr] & \\ & \text{Bank: } X_0 - \Delta_0 S_0 \end{tikzcd}$$

$$X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) = V_1$$

$$\begin{cases} V_1(H) = \Delta_0 \cdot uS_0 + (1+r)(X_0 - \Delta_0 S_0) & (1) \\ V_1(T) = \Delta_0 \cdot dS_0 + (1+r)(X_0 - \Delta_0 S_0) & (2) \end{cases}$$

$$(1)-(2) \Rightarrow V_1(H) - V_1(T) = \Delta_0 \cdot (u-d) S_0 \Rightarrow \Delta_0 = \frac{V_1(H)-V_1(T)}{(u-d)S_0}$$

$$V_1(H) = \frac{V_1(H)-V_1(T)}{(u-d)\cancel{S_0}} \cdot u\cancel{S_0} + (1+r)X_0 - (1+r)\frac{V_1(H)-V_1(T)}{(u-d)\cancel{S_0}} \cdot \cancel{S_0}$$

$$V_1(H) - \frac{V_1(H)-V_1(T)}{u-d} u + (1+r)\frac{V_1(H)-V_1(T)}{u-d} = (1+r)X_0$$

$$\Rightarrow X_0 = \frac{1}{1+r}\left(\frac{(u-d)V_1(H) - V_1(H)u + V_1(T)u + (1+r)V_1(H) - (1+r)V_1(T)}{u-d}\right)$$

$$X_0 = \frac{1}{1+r}\left(\underbrace{\frac{1+r-d}{u-d}}_{\tilde{p}_1} V_1(H) + \underbrace{\frac{u-(1+r)}{u-d}}_{\tilde{q}_1} V_1(T)\right).$$

- $\Omega = \{(w_1, \cdots, w_N) \mid w_i = \pm 1,\ 1 \le i \le N\}$.

$P : \Omega \to [0,1]$, $\sum_{w \in \Omega} p(w) = 1$ PMF.

$p_1 \in (0,1)$, $w = (w_1, \cdots, w_N)$,

$P(w) = p_1^{H(w)} q_1^{T(w)}$, $H(w) = \#$ of heads in w

$T(w) = \#$ of tails in w.

$$\sum_{w\in\Omega} p(w) = \sum_{w\in\Omega} p_1^{H(w)} q_1^{T(w)} = \sum_{k=0}^{N} \sum_{w\in\Omega_k} p_1^{H(w)} q_1^{T(w)} = \sum_{k=0}^{N} \underline{\sum_{w\in\Omega_k} p_1^{k} q_1^{N-k}}$$

$$= \underline{\sum_{k=0}^{N} \binom{N}{k} p_1^k q_1^{N-k}} = (p_1 + q_1)^N = 1.$$

For $k = 0, 1, \cdots, N$,

$\Omega_k = \{w \in \Omega \mid H(w) = k\}$.

$\underline{w} = (w_1, \cdots, w_N)$

$$A_i = \{ w \in \Omega \mid w_i = 1 \}, \quad 1 \le i \le N$$

$$\mathbb{P}(A_i) = \sum_{w \in A_i} p(w) = \sum_{k=1}^{N} \sum_{w \in B_k} p_1^{H(w)} q_1^{T(w)} = \sum_{k=1}^{N} \sum_{w \in B_k} p_1^{k} q_1^{N-k} = \sum_{k=1}^{N} \binom{N-1}{k-1} p_1^{k} q_1^{N-k}$$

$$B_k = \{ w \in A_i \mid H(w) = k \}, \quad 1 \le k \le N$$

$$k' = k-1$$

$$= \sum_{k'=0}^{N-1} \binom{N-1}{k'} p_1^{k'+1} q_1^{N-k'-1} = p_1 \sum_{k'=0}^{N-1} \binom{N-1}{k'} p_1^{k'} q_1^{(N-1)-k'} = p_1 (p_1 + q_1)^{N-1} = p_1$$

- Expectation.

$$X : \Omega \to \mathbb{R} \qquad \mathbb{E}X = \overbrace{\sum_{w \in \Omega} X(w) p(w)}^{\mathrm{I}} = \overbrace{\sum_{x \in \mathrm{Range}(X)} x \, \mathbb{P}(X = x)}^{\mathrm{II}}.$$

$$\mathrm{Range}(X) = \{ x_1, x_2, \cdots, x_k \}. \quad \Omega_j = \{ w \in \Omega \mid X(w) = x_j \}, \quad 1 \le j \le k.$$

$$\mathrm{I} = \sum_{j=1}^{k} \sum_{w \in \Omega_j} X(w) p(w) = \sum_{j=1}^{k} \sum_{w \in \Omega_j} x_j p(w) = \sum_{j=1}^{k} x_j \sum_{w \in \Omega_j} p(w) = \sum_{j=1}^{k} x_j \mathbb{P}(\Omega_j)$$

$$= \sum_{j=1}^{k} x_j \mathbb{P}(X = x_j)$$

$$= \mathrm{II}.$$

$N = 10 \qquad X_5 = \begin{cases} 1, & \text{if } \omega_5 = 1 \\ -1, & \text{if } \omega_5 = -1 \end{cases}$

~~Pf~~

$$\mathbb{E}X = \sum_{\omega \in \Omega} X(\omega)\, p(\omega) = 1 \cdot \mathbb{P}(X_5 = 1) + (-1) \cdot \mathbb{P}(X_5 = -1) = p_1 - q_1$$

$\uparrow$

2^{10} terms

- Independence

2 events A, B, $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$.

n events $A_1, A_2, \cdots, A_n$,

$\mathbb{P}(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \cdots \cdot \mathbb{P}(A_n)$

✓ $\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i) \cdot \mathbb{P}(A_j)$, $1 \le i < j \le n$

$\forall\ 1 \le i_1 < i_2 < \cdots < i_k \le n$

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \mathbb{P}(A_{i_1}) \cdot \mathbb{P}(A_{i_2}) \cdot \cdots \cdot \mathbb{P}(A_{i_k}).$$

$1 \le i < j \le n$, $\underline{\mathbb{P}(A_i \cap A_j)} = \underline{\mathbb{P}(A_i)} \cdot \underline{\mathbb{P}(A_j)} = \underline{p_1^2}$

$B_k = \{\omega \in A_i \cap A_j \mid H(\omega) = k\}$, $2 \le k \le N$.

$$\underline{\mathbb{P}(A_i \cap A_j)} = \sum_{w \in A_i \cap A_j} p(w) = \sum_{k=2}^{N} \sum_{w \in B_k} p_1^{H(w)} q_1^{T(w)} = \sum_{k=2}^{N} \sum_{w \in B_k} \underline{p_1^k q_1^{N-k}}$$

$$= \sum_{k=2}^{N} \binom{N-2}{k-2} p_1^k q_1^{N-k} \underset{k'=k-2}{=} \sum_{k'=0}^{N-2} \binom{N-2}{k'} p_1^{k'+2} q_1^{N-k'-2}$$

$$= p_1^2 \sum_{k'=0}^{N-2} p_1^{k'} q_1^{(N-2)-k'} = p_1^2 (p_1 + q_1)^{N-2} = p_1^2 = \mathbb{P}(A_i)\mathbb{P}(A_j)$$

- Conditional expectation

$\Omega = \{ w = (w_1, \cdots, w_N) \mid w_i = \pm 1, 1 \le i \le N \}$. X is a r.v.

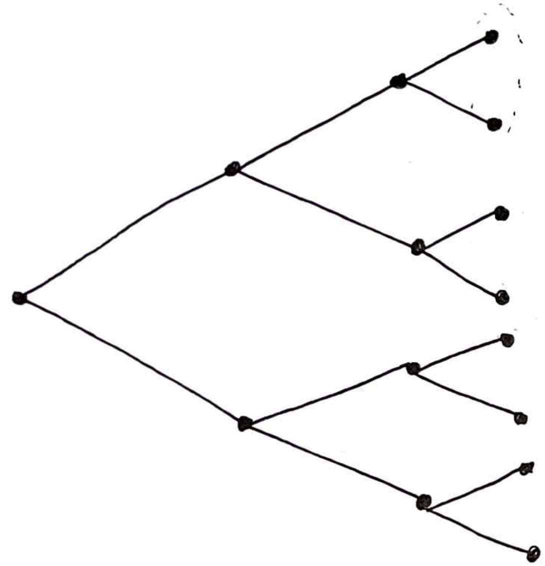
$1 \le n \le N$ $\quad \mathbb{E}_n X : \Omega \to \mathbb{R}$.

$\mathbb{E}(X \mid A)$ $\quad \mathbb{E}_n X(w) = \sum_{x \in Rang(X)} x \, \mathbb{P}(X = x \mid \Pi_n(w))$

$$\Pi_n(w) = \{ w' \in \Omega \mid w_1' = w_1, \cdots, w_n' = w_n \}.$$

$N=3$, $p_1 = \frac{2}{3}$, $q_1 = \frac{1}{3}$ ~~$\mathbb{E}_n$~~ $\mathbb{E}_n X(\omega) = \sum_{w \in Range(X)} x \, \mathbb{P}(X = x \mid \Pi_n(\omega))$

$\Pi_n(\omega) = \{\omega' \in \Omega \mid \omega_1' = \omega_1, \cdots, \omega_n' = \omega_n\}$.



ω	$P(\omega)$	X	Y	Z	$\mathbb{E}_2 X$	$\mathbb{E}_2 Y$	$\mathbb{E}_2 Z$	$\mathbb{E}_1 X$	$\mathbb{E}_1(\mathbb{E}_2 X)$
(1,1,1)	8/27	1	1	1	4/3	1	1/3	2	2
(1,1,−1)	4/27	2	1	−1	4/3	1	1/3	2	2
(1,−1,1)	4/27	3	2	1	10/3	2	1/3	2	2
(1,−1,−1)	2/27	4	2	−1	10/3	2	1/3	2	2
(−1,1,1)	4/27	5	3	1	16/3	3	1/3	6	6
(−1,1,−1)	2/27	6	3	−1	16/3	3	1/3	6	6
(−1,−1,1)	2/27	7	4	1	22/3	4	1/3	6	6
(−1,−1,−1)	1/27	8	4	−1	22/3	4	1/3	6	6

$n=0$ $\quad n=1 \quad n=2 \quad n=3$

$$\mathbb{E}X = \frac{1}{27}(8\times1 + 4\times2 + 4\times3 + 2\times4 + 4\times5 + 2\times6 + 2\times7 + 1\times8)$$
$$= 10/3$$

$$\mathbb{E}Y = \frac{1}{27}\left((8+4)\times1 + (4+2)\times2 + (4+2)\times3 + (2+1)\times4\right) = 2$$

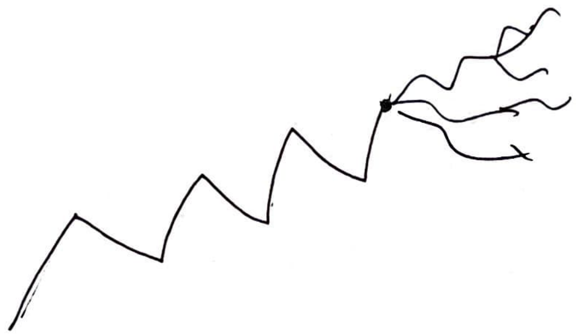
$$Z = \begin{cases} 1, & \omega_3 = 1 \\ -1, & \omega_3 = -1 \end{cases}$$

$$\mathbb{E}Z = \frac{1}{27}(8 - 4 + 4 - 2 + 4 - 2 + 2 - 1) = \frac{1}{3}.$$

- Z and $\mathcal{F}_2$ are independent
- Y is $\mathcal{F}_2$-measurable.

XY	$\mathbb{E}_2(XY) = Y \cdot \mathbb{E}_2 X$
1	4/3
2	4/3
6	20/3
8	20/3
15	16
18	16
28	88/3
32	88/3

$\omega = (\omega_1, \omega_2, \cdots, \omega_N)$



$\sigma(X) \subseteq \mathcal{P}(\Omega)$

Ω

$\{ \phi, \Omega, \cancel{\{HH\}}, \cancel{\{HT\}}, \underline{\{HH\} \cup \{HT\}}, \underline{\{TH\} \cup \{TT\}} \}$
$\cancel{\{TH\}}, \cancel{\{TT\}}$.

$\sigma(X) = \{ \underline{\{\omega \in \Omega \mid X(\omega) \in B\}} \mid B \subseteq \mathbb{R} \}$.

$$\mathbb{E}_2 X(1,1,1) = \sum_{k=1}^{8} k\, \mathbb{P}\left(X=k \mid \underset{\Pi_2(1,1,-1)}{\underline{\Pi_2(1,1,1)}}\right) = 1 \times \frac{2}{3} + 2 \times \frac{1}{3} = \frac{4}{3}$$

$$\Pi_2(1,1,1) = \{(1,1,1), (1,1,-1)\}.$$

$$\mathbb{P}(X=k \mid \{(1,1,1), (1,1,-1)\}) = \frac{\mathbb{P}(\{X=k\} \cap \{(1,1,1), (1,1,-1)\})}{\mathbb{P}(\{(1,1,1), (1,1,-1)\})}.$$

$$= \frac{\frac{8}{27}}{\frac{8}{27} + \frac{4}{27}} = \frac{2}{3}, \quad k=1$$

$$\frac{\frac{4}{27}}{\frac{8}{27} + \frac{4}{27}} = \frac{1}{3}, \quad k=2$$

$$\Pi_2(1,1,-1) = \Pi_2(1,1,1)$$

$$\Pi_2(1,-1,1) = \{(1,-1,1), (1,-1,-1)\} = \Pi_2(1,-1,-1).$$

$$\mathbb{E}_2 X(1,-1,+1) = \sum_{k=1}^{8} k\, \frac{\mathbb{P}(\{X=k\} \cap \Pi_2(1,-1,+1))}{\mathbb{P}(\Pi_2(1,-1,+1))}. = 3 \times \frac{\frac{4}{27}}{\frac{4}{27} + \frac{2}{27}} + 4 \times \frac{\frac{2}{27}}{\frac{4}{27} + \frac{2}{27}}$$

$$= 10/3$$

$$\mathbb{E}_2 Y(1,1,1) = \underbrace{Y(1,1,1)}_{=1} \times \frac{\frac{8}{27}}{\frac{8}{27}+\frac{4}{27}} + \underbrace{Y(1,1,-1)}_{=1} \times \frac{\frac{4}{27}}{\frac{8}{27}+\frac{4}{27}} = 1$$

$$\mathbb{E}_2 Z(1,1,1) = 1 \times \frac{2}{3} + (-1) \times \frac{1}{3} = \frac{1}{3}$$

$$\mathbb{E}_1 X(1,1,1) = \sum_{k=1}^{8} k\, \mathbb{P}(X=k \mid \Pi_1(1,1,1))$$

$$= 1 \times \frac{8}{8+4+4+2} + 2 \times \frac{4}{8+4+4+2} + 3 \times \frac{4}{8+4+4+2} + 4 \times \frac{2}{8+4+4+2}$$

$$= 2$$

$$\mathbb{E}_1(\mathbb{E}_2 X)(1,1,1) = \frac{4}{3} \times \frac{8}{18} + \frac{4}{3} \times \frac{4}{18} + \frac{10}{3} \times \frac{4}{18} + \frac{10}{3} \times \frac{2}{18} = 2$$

- $\mathbb{E}_n X$ is $\mathcal{F}_n$-measurable, $\forall\ 1 \le n \le N$.

- If X is $\mathcal{F}_n$-measurable, then $\mathbb{E}_n X = X$.

- If X and $\mathcal{F}_n$ are independent, then $\mathbb{E}_n X = \mathbb{E} X$.

- $1 \le m \le n \le N$

  $$\mathbb{E}_m(\mathbb{E}_n X) = \underbrace{\mathbb{E}_m X = \mathbb{E}_n(\mathbb{E}_m X)}$$

- If Y is $\mathcal{F}_n$-measurable, then $\mathbb{E}_n(XY) = Y\,\mathbb{E}_n X$.

- X and $\mathcal{F}_n$ independent, Y is $\mathcal{F}_n$-measurable

  $$\mathbb{E}_n(XY) = Y\,\mathbb{E}_n X = Y\,\mathbb{E} X.$$