2. Syllabus Overview

- Class website and full syllabus: https://www.math.cmu.edu/~gautam/sj/teaching/2021-22/944-scalc-finance1
- TA's: Shukun Long <shukunl@andrew.cmu.edu>.
- Homework Due: 10:10AM Oct 28, Nov 4, 11, 23, 30, Dec 7
- Midterm: Tue, Nov 16, in class (May be delayed to Nov 18 if we have not covered Itô's formula in time.)
- **Homework:**
  - ▷ Good quality scans please! Use a scanning app, and not simply take photos. (I use Adobe Scan.)
  - ▷ 20% penalty if turned in within an hour of the deadline. 100% penalty after that.
  - ▷ One homework assignments can be turned in 24h late without penalty.
  - ▷ Bottom homework score is dropped from your grade (personal emergencies, interviews, other deadlines, etc.).
  - ▷ Collaboration is encouraged. Homework is not a test – ensure you learn from doing the homework.
  - ▷ You must write solutions independently, and can only turn in solutions you fully understand.
- **Academic Integrity**
  - ▷ Zero tolerance for violations (automatic **R**).
  - ▷ Violations include:
    - Not writing up solutions independently and/or plagiarizing solutions
    - Turning in solutions you do not understand.
    - Seeking, receiving or providing assistance during an exam.
  - ▷ All violations will be reported to the university, and they may impose additional penalties.
- **Grading:** 10% homework, 30% midterm, 60% final.

Course Outline.

- Review of Fundamentals: Replication, arbitrage free pricing.
- Quick study of the multi-period binomial model.
  - ▷ Simple example of replication / arbitrage free pricing.
  - ▷ Understand conditional expectations. (Have an explicit formula.)
  - ▷ Understand measurablity / adaptedness. (Can be stated easily in terms of coin tosses that have / have not occurred.)
  - ▷ Understand risk neutral measures. Explicit formula!
- Develop tools to price securities in continuous time.
  - ▷ Brownian motion (not as easy as coin tosses)
  - ▷ Conditional expectation: No explicit formula!
  - ▷ Itô formula: main tool used for computation. Develop some intuition.
  - ▷ Measurablity / risk neutral measures: much more abstract. Complete description is technical. But we need a working knowledge.
  - ▷ Derive and understand the Black-Scholes formula.

3. Replication and Arbitrage

3.1. Replication and arbitrage free pricing.

- Start with a *financial market* consisting of traded assets (stocks, bonds, money market, options, etc.)
- We model the price of these assets through random variables (stochastic processes).
- **No Arbitrage Assumption:**
  - ▷ In order to make money, you have to take risk. (Can't make something out of nothing.)
  - ▷ Mathematically: For any trading strategy such that $X_0 = 0$, and $X_n \geqslant 0$, you must also have $X_n = 0$ almost surely.
  - ▷ Equivalently: There doesn't exist a trading strategy with $X_0 = 0$, $X_n \geqslant 0$ and $\boldsymbol{P}(X_n > 0) > 0$.
- Now consider a non-traded asset Y (e.g. an option). How do you price it?
- *Arbitrage free price:* If given the opportunity to trade Y at price V_0, the market remains arbitrage free, then we say V_0 is the arbitrage free price of Y.

Bank ← interest rate r.

Stocks/Bonds etc.

Market

No Arb: To make \$ you have to take risk.

If Say $X_0 = 0$ & $X_n \geqslant 0$

X_0 ~ initial wealth.

X_n ~ Wealth at time n.

Then: $X_n = 0$

Arbitrage free Price.

Market.

$Y \rightarrow$ Non traded asset.

(e.g. Call option).

AFP: If given the opportunity to trade the NT asset at price V_0 the market remains arb free, then we call V_0 = the arb free price.

- *We will almost always find the arbitrage free price by* **replication**.
  - ▷ Say the non-traded asset pays V_N at time N (e.g. call options).
  - ▷ Try and *replicate the payoff*:
    - Start with X_0 dollars.
    - Use only traded assets and ensure that at maturity $X_N = V_N$.
  - ▷ Then the arbitrage free price is uniquely determined, and must be X_0.

Remark 3.1. The arbitrage free price is *unique* if and only if there is a replicating strategy! In this case, the arbitrage free price is exactly the initial capital of the replicating strategy.

Find AFP by $\boxed{\text{Replication.}}$

$V_N \longrightarrow$ Payoff at time N.

Replication: (1) Start with X_0 \$.

(2) Use only traded assets.

(3) Goal End with $\underbrace{X_N}_{\text{Wealth at time } N.} = V_N.$

Replicate a Security.

If you replicate a security with some trading strat.

X_0 = initial wealth

X_1 = wealth at time 1

$\vdots$

X_N = " " " N. $\quad (X_N \overset{\text{(Replication)}}{=} V_N)$.

V_N ← payoff.

$\boxed{\text{Then AFP of the security must be } X_0.}$

3.2. **Example: One period Binomial model.**

- Consider a market with a stock, and money market account.
- Interest rate for borrowing and lending is r. No transaction costs. Can buy and sell fractional quantities of the stock.
- *Model assumption:* Flip a coin that lands heads with probability $p_1 \in (0,1)$ and tails with probability $q_1 = 1 - p_1$. Model $S_1 = uS_0$ if heads, and $S_1 = dS_0$ if tails.
  - ▷ S_0 is stock price at time 0 (known).
  - ▷ S_1 is stock price after one time period (random).
  - ▷ u, d are model parameters (pre-supposed). Called the up and down factors. (Will always assume $0 < d < u$.)

Proposition 3.2. *There's no arbitrage in this model if and only if $d < 1 + r < u$.*

Proof:

M.M ← interest rate $r > -1$

Stock:

Market.

$u, d, p, q \to$ Model parameters.

Model assumption.

S_0 = price at time 0 of stock.

$$\begin{tikzcd}[column sep=large] & uS_0 \\ S_0 \arrow[ur, "p"] \arrow[dr, "q"'] & \\ & dS_0 \end{tikzcd}$$

S_1

$$S_1 = \begin{cases} uS_0 & \text{if heads.} \\ dS_0 & \text{if tails.} \end{cases}$$

Remark: No arb in this model

$\iff d < 1+r < u.$

Intuition: (1) If $\underbrace{d \geq 1+r}$. $\longrightarrow$ Bank: $X_0 \longrightarrow (1+r)X_0$.

Stock: $S_0 \longrightarrow \underbrace{\begin{cases} uS_0 \\ dS_0 \end{cases}}$

$\underbrace{\geq (1+r)S_0}$

Arb Opportunity!

(2) If $1+r \geq u \to$ Reverse.

(3) Calc & check that

$d < 1+r < u \Rightarrow$ No arb.

Proposition 3.3. *Say a security pays V_1 at time 1 (V_1 can depend on whether the coin flip is heads or tails). The arbitrage free price at time 0 is given by*

$$\longrightarrow \quad V_0 = \frac{1}{1+r}(\tilde{p}_1 V_1(H) + \tilde{q}_1 V_1(T)) = \frac{1}{1+r}\tilde{\boldsymbol{E}}V_1 , \qquad \textit{where } \tilde{p}_1 = \frac{1+r-d}{u-d}, \quad \tilde{q} = \frac{u-(1+r)}{u-d}.$$

The replicating strategy holds $\underline{\Delta_0 = \dfrac{V_1(H) - V_1(T)}{(u-d)S_0}}$ *shares of stock at time* 0.

Proof:

(Assume no arb).

Security pay V_1 at time 1

($V_1 \longrightarrow$ can depend on outcome of first coin toss.)

Claim: AFP at time 0: $V_0 = \dfrac{1}{1+r}\left(\underline{\tilde{p}}\, V_1(H) + \underline{\tilde{q}}\, V_1(T)\right)$.

$$\tilde{p} = \frac{1+r-d}{u-d} \qquad \& \ \tilde{q} = \frac{u-(1+r)}{u-d}.$$

Reasons: Try & replicate V_1

Start with X_0 \$ → Cash. $(X_0 - \Delta_0 S_0)$.

→ Stock. Δ_0 shares.

X_1 = wealth at time 1 $= \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0)$.

Δ_0: # shares at time 0

S_1: new price of stock.

Replication: Want $X_1 = V_1$ weather heads or tails!

$$V_1(H) = X_1(H) = \Delta_0 (u S_0) + (1+r)(X_0 - \Delta_0 S_0) \quad \text{(if heads)}.$$

$$V_1(H) = X_1(T) = \Delta_0 (d S_0) + (1+r)(X_0 - \Delta_0 S_0) \quad \text{(if tails)}.$$

2 Eqns. (linear)

2 Unknowns. $(X_0$ & $\Delta_0)$. } Solve $\longrightarrow$ Gives the formula.

4. **Multi-Period Binomial Model.**

- Same setup as the one period case $0 < d < 1 + r < u$, and toss coins that land heads with probability p_1 and tails with probability q_1.
- Except now the security matures at time $N > 1$.
- Stock price: $S_{n+1} = uS_n$ if $n + 1$-th coin toss is heads, and $S_{n+1} = dS_n$ otherwise.
- To replicate it a security, we start with capital X_0.
- Buy Δ_0 shares of stock, and put the rest in cash.
- Get $X_1 = \Delta_0 S_1 + (1 + r)(X_0 - \Delta_0 S_0)$.
- Repeat. *Self Financing Condition:* $X_{n+1} = \Delta_n S_{n+1} + (1 + r)(X_n - \Delta_n S_n)$.
- *Adaptedness:* Δ_n can only depend on outcomes of coin tosses before n!

Coin flip → Heads → multiply stock price by u.
↳ Tails → " " " " d.

Security matures after N time periods ($N > 1$.

$u^2 S_0 = S_2$

$S_1 = uS_0$

S_0

$udS_0 = S_2$

$S_1 = dS_0$

$d^2 S_0 = S_2$.

Wealth evolution :

$X_0 \to$ initial wealth.

$\Delta_0 \to$ # shares of stock bought at time 0.

$X_1 \to$ wealth at time 1 : $X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0)$.

At time 1 : Change pos. $\to$ hold Δ_1 shares of stock.

$$X_2 = \Delta_1 S_2 + (1+r)(X_1 - \Delta_1 S_1)$$

$\vdots$

$$\boxed{X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n).}$$

Self financing.
(No external cash flow)

<u>Adaptedness</u>: Δ_n can only use outcomes of coin tosses before (or at) time n.

$\rightarrow$ NOT outcomes of coin tosses after <u>time n</u>.

Proposition 4.1. *Consider a security that pays V_N at time N. Then for any $n \leqslant N$:*

$$V_n = \underbrace{\frac{1}{(1+r)} \tilde{\mathbf{E}}_n V_N}\,, \qquad \underbrace{\Delta_n = \frac{V_{n+1}(\omega_{n+1} = H) - V_{n+1}(\omega_{n+1} = T)}{(u-d)S_n}}\,.$$

- *V_n is the arbitrage free price at time $n \leqslant N$.*
- *Δ_n is the number of shares held in the replicating portfolio at time n (trading strategy).*

Question 4.2. *Why does this work?* ←

Question 4.3. *What is $\tilde{\mathbf{E}}_n$? (It's different from $\mathbf{E}$, and different from $\mathbf{E}_n$).*

A security with payoff V_N can be replicated.

~~APF~~ AFP at time $n \leq N$

$= $ Wealth of Rep port at time n

$$= \frac{1}{(1+r)^{N-n}} \cdot \underbrace{\tilde{E}_n}_{\text{IOU}} V_N .$$

↑

IOU.

4.1. **Quick review probability (finite Sample spaces).** This is just a quick reminder, to fix notation. Read one of the references, or look over the prep material / videos for a more through treatment. The only thing we will cover in any detail is conditional expectation.

Let $N \in \mathbb{N}$ be large (typically the maturity time of financial securities).

Definition 4.4. The *sample space* is the set $\Omega = \{(\omega_1, \dots, \omega_N) \mid \text{each } \omega_i \text{ represents the outcome of a coin toss.}\}$

▷ E.g. $\omega_i \in \{H, T\}$, or $\omega_i \in \{\pm 1\}$. (Each ω_i could also represent the outcome of the roll of a M sided die.)

Definition 4.5. A *sample point* is a point $\omega = (\omega_1, \dots, \omega_N) \in \Omega$.

▷ Each sample point represents the outcome of a sequence of *all* coin tosses from 1 to N.

Definition 4.6. A *probability mass function* is a function $p\colon \Omega \to [0,1]$ such that $\sum_{\omega \in \Omega} p(\omega) = 1$.

Example 4.7. Typical example: Fix $p_1 \in (0,1)$, $q_1 = 1 - p_1$ and set $p(\omega) = p_1^{H(\omega)} q_1^{T(\omega)}$. Here $H(\omega)$ is the number of heads in the sequence $\omega = (\omega_1, \dots, \omega_N)$, and $T(\omega)$ is the number of tails.

Definition 4.8. An event is a subset of Ω. Define $\boldsymbol{P}(A) = \sum_{\omega \in A} p(\omega)$.

Example 4.9. $A\{\omega \in \Omega \mid \omega_1 = +1\}$. Check $\boldsymbol{P}(A) = p_1$.

+1 = Heds, −1 = Tails.

outcome of a cointoss.

$\Omega \to$ sample space

$= \{(\omega_1, \omega_2, \dots \omega_N) \mid \omega_i \in \{\pm 1\}\}$

$\omega = (\omega_1, \dots \omega_N) \in \Omega \leftarrow$ sample point.

PMF: $p : \Omega \to [0,1]$. & $\sum_{\omega \in \Omega} p(\omega) = 1$.

$p(\omega) \approx$ prob that this particular seq of occurs.

$\underline{\underline{A}} \subseteq \Omega$ (any subset) $\to$ called an event.

<u>Eg:</u> fix $p_1 \in (0,1)$, $q_1 = 1 - p_1$

$$p(\omega) = p_1^{H(\omega)} q_1^{T(\omega)}$$

$H(\omega)$ = # heals in the seq $(\omega_1, \dots \omega_N)$.

$T(\omega)$ = " tails " " " "

$P(A)$ = prob that the ent A occurs $= \sum_{\omega \in A} p(\omega)$.

4.2. **Random Variables and Independence.**

Definition 4.10. A *random variable* is a function $X : \Omega \to \mathbb{R}$.

Example 4.11. $X(\omega) = \begin{cases} 1 & \omega_2 = +1\,, \\ -1 & \omega_2 = -1\,, \end{cases}$ is a random variable corresponding to the outcome of the second coin toss.

A Random Var. is fn $X : \Omega \longrightarrow \mathbb{R}$.

Definition 4.12. The *expectation* of a random variable X is $\boldsymbol{E}X = \sum X(\omega)p(\omega)$.

Remark 4.13. Note if $\text{Range}(X) = \{x_1, \ldots, x_n\}$, then $\boldsymbol{E}X = \sum X(\omega)p(\omega) = \sum_1^n x_i \boldsymbol{P}(X = x_i)$.

Definition 4.14. The *variance* of a random variable is $\text{Var}(X) = \boldsymbol{E}(X - \boldsymbol{E}X)^2$.

Remark 4.15. Note $\text{Var}(X) = \boldsymbol{E}X^2 - (\boldsymbol{E}X)^2$.

$$\mathbb{E}X = \text{"mean"} = \text{"avge of } X\text{"}.$$

$$= \sum X(\omega)\, p(\omega)$$

$$= \sum_{x_i \in \text{Range}(X)} x_i\, P(X = x_i).$$

Definition 4.16. Two events are independent if $\boldsymbol{P}(A \cap B) = \boldsymbol{P}(A)\boldsymbol{P}(B)$.

Definition 4.17. The events $A_1, \ldots, A_n$ are independent if for any sub-collection $A_{i_1}, \ldots, A_{i_k}$ we have

$$\boldsymbol{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \boldsymbol{P}(A_{i_1})\boldsymbol{P}(A_{i_2}) \cdots \boldsymbol{P}(A_{i_k}).$$

Remark 4.18. When $n > 2$, it is not enough to only require $\boldsymbol{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \boldsymbol{P}(A_1)\boldsymbol{P}(A_2) \cdots \boldsymbol{P}(A_n)$

Definition 4.19. Two random variables are independent if $\boldsymbol{P}(X = x,\ Y = y) = \boldsymbol{P}(X = x)\boldsymbol{P}(Y = y)$ for all $x, y \in \mathbb{R}$.

Definition 4.20. The random variables $X_1, \ldots, X_n$ are independent if for all $x_1, \ldots, x_n \in \mathbb{R}$ we have

$$\boldsymbol{P}(X_1 = x_1,\ X_2 = x_2, \ldots, X_n = x_n) = \boldsymbol{P}(X_1 = x_1)\boldsymbol{P}(X_2 = x_2) \cdots \boldsymbol{P}(X_n = x_n)\,.$$

Remark 4.21. Independent random variables are uncorrelated, but not vice versa.

4.3. Filtrations.

Definition 4.22. We define a *filtration* on Ω as follows:

- $\triangleright$ $\mathcal{F}_0 = \{\emptyset, \Omega\}$.
- $\triangleright$ $\mathcal{F}_1$ = all events that can be described by only the first coin toss. E.g. $A = \{\omega \mid \omega_1 = +1\} \in \mathcal{F}_1$.
- $\triangleright$ $\mathcal{F}_n$ = all events that can be described by only the first n coin tosses. E.g. $A = \{\omega \mid \omega_1 = 1, \omega_3 = -1, \omega_n = 1\} \in \mathcal{F}_n$.

Remark 4.23. Note $\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_N = \mathcal{P}(\Omega)$.

Remark 4.24. If $A, B \in \mathcal{F}_n$, then so do A^c, B^c, $A \cap B$, $A \cup B$, $A - B$, $B - A$.

$\mathcal{F}_0 = \{\phi, \Omega\} \leftarrow$ "info you have before tossig any coins".

$\mathcal{F}_1$ = All events. that can be descrbed using ONLY. the first coin toss. (E.g. $A = \{1^{st}$ toss is heads$\} \in \mathcal{F}_1$

$\cancel{A}B = \{2^{nd}$ toss is tails$\} \notin \mathcal{F}_1$.

$\vdots$

$\mathcal{F}_n$ = All evts that can be desc using only the first n coin tosses.

Definition 4.25. Let $n \in \{0, \dots, N\}$. We say a random variable X is $\mathcal{F}_n$-measurable if $X(\omega)$ only depends on $\omega_1, \dots, \omega_n$.

▷ Equivalently, for any $B \subseteq \mathbb{R}$, the event $\{X \in B\} \in \mathcal{F}_n$.

Remark 4.26 (Use in Finance). For every n, the trading strategy at time n (denoted by Δ_n) must be $\mathcal{F}_n$ measurable. We can not trade today based on tomorrows price.

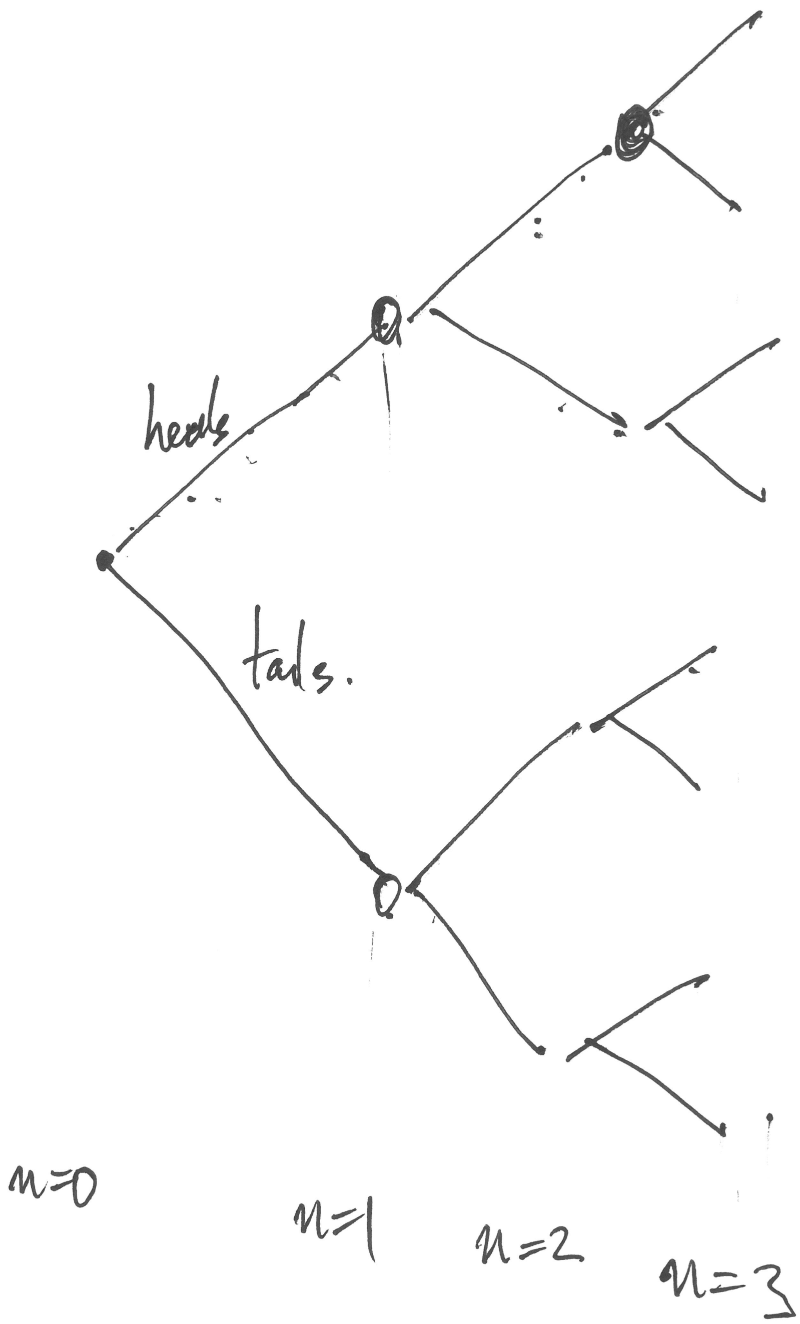
Example 4.27. If we represent Ω as a tree, $\mathcal{F}_n$ measurablity can be visualized by checking constancy on leaves.

① X is $\mathcal{F}_n$ meas if $X(\omega)$ only depends on $\omega_1, \omega_2 \cdots \omega_n$ & not $\omega_{n+1} \cdots \omega_N$.

② X is $\mathcal{F}_n$-meas. $\Longleftrightarrow$ For any $B \subseteq \mathbb{R}$, $\{X \in B\} \in \mathcal{F}_n$.

③ Finance: $\Delta_n \rightarrow$ always has to be $\mathcal{F}_n$-meas.

Ω.



X	Y
1	5
3	5
5	6
7	6
11	8
13	8
17	10
19	10

Y is $\mathcal{F}_2$ meas.

Q: Is X $\mathcal{F}_2$-meas.

(No).

X = 7 if toss H, T, T

4.4. **Conditional expectation.**

Definition 4.28. Let X be a random variable, and $n \leqslant N$. We define $\boldsymbol{E}(X \mid \mathcal{F}_n) = \boldsymbol{E}_n X$ to be the *random variable* given by

$$\boldsymbol{E}_n X(\omega) = \sum_{x_i \in \mathrm{Range}(X)} x_i \boldsymbol{P}(X = x_i \mid \Pi_n(\omega)), \qquad \text{where} \qquad \Pi_n(\omega) = \{\omega' \in \Omega \mid \omega'_1 = \omega_1, \ \ldots, \omega'_n = \omega_n\}$$

Remark 4.29. $\boldsymbol{E}_n X$ is the "best approximation" of X given only the first n coin tosses.

Remark 4.30. The above formula does not generalize well to infinite probability spaces. We will develop certain properties of $\boldsymbol{E}_n$, and then only use those properties going forward.

Example 4.31. If we represent Ω as a tree, $\boldsymbol{E}_n X$ can be computed by averaging over leaves.

$E_n X(\omega) =$ cond exp of X given $\mathcal{F}_n$.

$$= E(X \mid \mathcal{F}_n)$$

$$= \sum_{x_i \in \mathrm{Range}(X)} x_i P(X = x_i \mid \Pi_n(\omega))$$

$$\Pi_n(\omega) = \{ \omega' \mid \omega_1 = \omega'_1, \ \omega_2 = \omega'_2 \ \cdots \ \omega_n = \omega'_n \}.$$