Stochastic Calculus for Finance I: Final.

2021-12-09

- *This is a closed book test. You may use a calculator. You may not give or receive assistance.*
- *Your calculator must not be able to access the internet, or store/read document files (PDF, word, etc.)*
- *You have 3 hours. The exam has a total of 8 questions and 40 points.*
- *The questions are roughly ordered by difficulty. Good luck.*

In this exam W always denotes a standard Brownian motion, and the filtration $\{\mathcal{F}_t \mid t \geqslant 0\}$ (if not otherwise specified) is the Brownian filtration.

[5] 1. Let $X_t = \dfrac{W_t^2}{1+t}$. Find functions $g = g(t,x)$ and $h = h(t,x)$ such that $X_t = \displaystyle\int_0^t f(s, W_s)\,ds + \int_0^t g(s, W_s)\,dW_s$.

[5] 2. Consider the N period binomial model with $0 < d < 1 + r < u$. A *variance swap* that expires at time N pays

$$V_N = \frac{1}{N} \sum_{n=0}^{N-1} \Big(\log\Big(\frac{S_{n+1}}{S_n}\Big)\Big)^2 - K\,,$$

where S_n is the stock price at time n, and K is the strike price. Let V_n be the arbitrage free price of this contract at time n. Find a formula for K in terms of the model parameters so that $V_0 = 1$.

[5] 3. Consider a market consisting of a money market account and a stock. The interest rate in the money market account is $r \geqslant 0$, and the price of the stock is modelled by a geometric Brownian motion with mean return rate α and volatility σ. A straddle option with strike $K > 0$ and maturity T pays the holder $|S_T - K|$ at time T. Find the arbitrage free price of this option at all times $t \in [0, T]$. Express your answer in terms of the model parameters $\alpha, r, \sigma, S, t, T, K$ without using expectations or integrals. Your answer may involve the cumulative distribution function of the normal distribution.

[5] 4. Let $X_t \stackrel{\text{def}}{=} W_t \displaystyle\int_0^t e^{-2s^2} W_s\,dW_s$. Find a function $f = f(t,x)$ so that the process $X_t - \displaystyle\int_0^t f(s, W_s)\,ds$ is a martingale.

[5] 5. Let X be a random variable such that $\boldsymbol{P}(X = 2) = 1/3$ and $\boldsymbol{P}(X = 3) = 2/3$. Let Y be a standard normal random variable that is independent of X. Find $\boldsymbol{E}(e^{XY} | X)$ and $\boldsymbol{E} e^{XY}$.

[5] 6. Given $0 \leqslant s \leqslant t$, find $\boldsymbol{E}_s\Big(W_t \displaystyle\int_0^t W_r\,dr\Big)$. Express the final answer without involving expectations.

[5] 7. Let X be an Itô process such that $dX_t = t^2\,dt + t\,dW_t$. Fix $T > 0$, and let Z_T be an $\mathcal{F}_T$-measurable random variable such that $Z_T > 0$ and $\boldsymbol{E} Z_T = 1$. Define a new measure $\tilde{\boldsymbol{P}}$ by $d\tilde{\boldsymbol{P}} = Z_T\,d\boldsymbol{P}$. Find a formula for Z_T so that the process X is a martingale under $\tilde{\boldsymbol{P}}$.

[5] 8. Consider a financial market consisting of a stock and a money market account. Suppose the money market account has a constant return rate r and the stock price is given by a stochastic process S such that

$$dS_t = \alpha_t S_t\,dt + \sigma_t S_t\,dW_t\,.$$

Here $\alpha = \alpha_t$ is an adapted process, and $\sigma = \sigma(t)$ is a given, non-random, function of t. Let $K, T > 0$ and consider a European call option on S with strike K and maturity T. Given $t \in [0, T)$, find the arbitrage free price of this option at time t. Your final answer may involve r, σ, t, T, K, $S(t)$, the cumulative distribution function of the standard normal, and Riemann integrals of powers of σ. Your final answer may not involve expectations, probabilities or other integrals. (That is integrals of the form $\int_{2t}^{T-t} \sigma^3(s)\,ds$ are OK, but $\int_{-\infty}^{x} e^{-ry+y^2}\,dy$ are not.)
Hint: Under risk neutral measure find a normally distributed random variable Y such that $S_T = S_t e^Y$, and Y is independent of S_t.