Continuous Time Finance: Midterm 2.

2022-04-06

- *This is a closed book test. You may not use phones, calculators, or other electronic devices.*
- *You may not give or receive assistance.*
- *You have 50 minutes. The exam has a total of 4 questions and 40 points.*
- *The questions are roughly ordered by difficulty. Good luck.*

In this exam W always denotes a standard Brownian motion, and the filtration $\{\mathcal{F}_t \mid t \geqslant 0\}$ is the Brownian filtration. Here are a few formulae that you can use:

- Solution formula to the Black Scholes PDE:

$$f(t,x) = \int_{-\infty}^{\infty} e^{-r\tau} g\Big(x \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)\tau + \sigma\sqrt{\tau}\, y\Big)\Big) \frac{e^{-y^2/2} dy}{\sqrt{2\pi}}\,, \qquad \tau = T - t\,.$$

- Black Scholes Formula for European calls, and the Greeks

$$c(t,x) = xN(d_+) - Ke^{-r\tau}N(d_-) \qquad d_\pm \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big)\,, \qquad N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy\,,$$

$$\partial_x c = N(d_+)\,, \qquad \partial_x^2 c = \frac{1}{x\sigma\sqrt{2\pi\tau}} \exp\Big(\frac{-d_+^2}{2}\Big)\,, \qquad \partial_t c = -rKe^{-r\tau}N(d_-) - \frac{\sigma x}{\sqrt{8\pi\tau}} \exp\Big(\frac{-d_+^2}{2}\Big)\,.$$

$\boxed{10}$ 1. Let S be a geometric Brownian motion with mean return rate α and volatility σ. Find $\boldsymbol{E}S_t$. Express your answer in terms of S_0, α, σ and t without using expectations or integrals. You may assume $S_0 > 0$ is not random.

[Even though this was a question on your homework, please provide a complete derivation here. You may use other standard results from class/homework that are independent of this problem.]

$\boxed{10}$ 2. Consider a market with a bank and a stock. The bank has interest rate r and the stock price is modelled by a geometric Brownian motion with mean return rate α and volatility σ. Let $0 < K_1 < K_2$, $T > 0$ and S_t denote the stock price at time t. Consider a security that pays $S_T - K_1$ if $S_T \in [K_1, K_2)$, pays $K_2 - K_1$ if $S_T \geqslant K_2$, and pays nothing otherwise. Find the arbitrage free price of this security at time $t \in [0, T]$. Express your answer in terms of $t, T, K_1, K_2, \alpha, \sigma, r, S_t$ and the CDF of the standard normal, without using expectations or integrals.

$\boxed{10}$ 3. Let B be a Brownian motion that is independent of W, and let $M_t = \int_0^t W_s B_s^2\, dW_s$. Find $\boldsymbol{E}(W_t^2 M_t)$. Express your answer in terms of t without using expectations or integrals.

$\boxed{10}$ 4. Consider a market with a bank and a stock. The bank has interest rate r, and the stock price satisfies

$$dS_t = b(t, S_t)\, dt + \sigma(t, S_t)\, dW_t\,,$$

where $b = b(t,x)$ and $\sigma = \sigma(t,x)$ are two given functions. You may assume $\sigma(t,x) > 0$ for all x, t. Given a function g, consider the security that pays $g(S_T)$ at maturity time T. Suppose this security is replicable, and the wealth of the replicating portfolio is at time t is $f(t, S_t)$ for some function f. Express $\partial_t f$ in terms of t, x, f, $\partial_x f$, $\partial_x^2 f$, b, σ and r. Your answer should not involve S_t, expectations or integrals.