HW 14 Q7: $R(t) = r_0 + \theta t + \kappa \tilde{B}_t$

$(\tilde{B} \to$ BM under RNM $\tilde{P})$

ⓐ Bond price: $B_{t,T}$

$$D_t = \exp\left(-r_0 t - \theta \frac{t^2}{2} - \kappa \int_0^t \tilde{B}_s \, ds\right)$$

$$B_{t,T} = \tilde{E}_t \left(\frac{D_T}{D_t} \right) = \tilde{E}_t \exp\left(-r_0 T - \frac{\theta}{2}(T^2 - t^2) - \kappa \int_t^T \tilde{B}_s \, ds \right)$$

$$= \exp\left(-r_0 T - \frac{\theta}{2}(T^2 - t^2) \right) \underbrace{\tilde{E}_t \exp\left(-\kappa \int_t^T \tilde{B}_s \, ds \right)}$$

Note: $\int_t^T \tilde{B}_s \, ds \sim N(0, \sigma^2)$ (e.g. $\int_0^T \tilde{B}_s \, ds \approx \underbrace{\sum \tilde{B}_{t_i} (t_{i+1} - t_i)}_{\text{Normal}}$)

Find σ^2

$$\sigma^2 = \tilde{E}\left(\int_t^T \tilde{B}_s \, ds\right)^2 = \tilde{E}\left(\int_t^T \tilde{B}_s \, ds\right)\left(\int_t^T \tilde{B}_r \, dr\right)$$

$$= \tilde{E} \int_t^T \int_t^T \tilde{B}_s \tilde{B}_r \, ds \, dr = \int_{s=t}^T \int_{r=t}^T (s \wedge r) \, ds \, dr$$

& solve.

u

Say $r \leq s$:

Note $\hat{\mathbb{E}}(\hat{B}_s \hat{B}_r) = \hat{\mathbb{E}}\,\hat{\mathbb{E}}_r(\hat{B}_s \hat{B}_r)$

$= \hat{\mathbb{E}}(\hat{B}_r \underbrace{\hat{\mathbb{E}}_r \hat{B}_s}_{})$

$= \hat{\mathbb{E}}(\hat{B}_r \hat{B}_r) = r$

$\left.\right\} \Rightarrow \boxed{\hat{\mathbb{E}}(\hat{B}_r \hat{B}_s) = r \wedge s}$

By symmetry if $r \geq s$ $\hat{\mathbb{E}}(\hat{B}_r \hat{B}_s) = s$

Now $\tilde{B}_t \exp\left(-\kappa\int_t^T \tilde{B}_s\,ds\right) = \tilde{B}_t \exp\left(-\kappa\int_t^T (\tilde{B}_s-\tilde{B}_t)\,ds - \kappa(T-t)\tilde{B}_t\right)$

$$= \exp\left(-\kappa(T-t)\tilde{B}_t\right)\,\tilde{E}_t \exp\left(-\kappa\int_t^T (\tilde{B}_s-\tilde{B}_t)\,ds\right)$$

$$= \exp\left(-\kappa(T-t)\tilde{B}_t\right)\,\tilde{E}\exp\left(-\kappa \underbrace{\int_t^T(\tilde{B}_s-\tilde{B}_t)\,ds}_{\sim N(0,\,\sigma^2)}\right)$$

Find σ^2 from above & use MGF to get $\tilde{E}\exp\left(-\kappa\int_t^T(\tilde{B}_s-\tilde{B}_t)\,ds\right)$

MGF: $\mathbb{E}\, e^{\lambda N(0,1)} = e^{\lambda^2/2}$

CF: $\mathbb{E}\, e^{i\lambda N(0,1)} = e^{-\lambda^2/2}$

$$\mathbb{E}\, e^{\lambda N(0,\sigma^2)} = \mathbb{E}\, e^{\lambda\sigma \underbrace{\frac{1}{\sigma} N(0,\sigma^2)}_{N(0,1)}}$$

$$= \mathbb{E}\, e^{\lambda\sigma N(0,1)} = e^{\frac{\lambda^2\sigma^2}{2}}$$

TYPO Prob 4.38 (a)

$$\boxed{F_{0 \to t_0,T} = \frac{S_t}{B_{t_0,T}}}$$

(Will fix online shortly)

Q7 b) $F_{0 \to t} = \frac{S_t}{B_{t,T}}$ & use $B_{t,T}$ from (a)

Q7 c) $Put_t = \hat{B}_t S_T$

$$dS_t = R_t S_t\,dt + \sigma S_t\,dB_t$$

Itô $$d(\ln S_t) = \frac{1}{S_t}\,dS_t - \frac{1}{2S_t^2}\,d[S,S]_t$$

$$= R_t\,dt + \sigma\,dB_t - \frac{\sigma^2}{2}\,dt$$

$$\Rightarrow \ln\left(\frac{S_t}{S_0}\right) = \int_0^t R_s\,ds - \frac{\sigma^2}{2}t + \sigma B_t$$

$$\Rightarrow S_t = S_0 \exp\left(\int_0^t R_s\,ds - \frac{\sigma^2}{2} t + \sigma \tilde{B}_t \right)$$

$$\& \quad S_T = S_t \exp\left(\int_t^T R_s\,ds - \frac{\sigma^2}{2}(T-t) + \sigma\left(\tilde{B}_T - \tilde{B}_t\right) \right)$$

$$Fwd_t = \tilde{E}_t S_T = S_t\, e^{-\frac{\sigma^2}{2}(T-t)}\, \underbrace{\tilde{E}_t \exp\left(\int_t^T R_s\,ds + \sigma\left(\tilde{B}_T - \tilde{B}_t\right) \right)}$$

$$\tilde{E}_t \exp\left(\int_t^T \left(r_0 + \theta s + \kappa \tilde{B}_s\right) ds + \overbrace{\sigma\left(\tilde{B}_T - \tilde{B}_t\right)}^{\text{Find the}} \right)$$

$$\exp\left(r_0(T-t) + \theta\frac{(T^2-t^2)}{2} \right) \hat{B}_t \exp\Bigg(\underbrace{\kappa\int_t^T (\hat{B}_s - \hat{B}_t)\, ds}_{\text{Normally dist \& indep of } \mathcal{F}_t} + \overbrace{\kappa(T-t)\hat{B}_t}^{\mathcal{F}_t \text{ meas}}$$

$$+ \underbrace{\sigma(\hat{B}_T - \hat{B}_t)}_{\text{Normally dist \& indep of } \mathcal{F}_t} \Bigg)$$

Normally dist & indep of $\mathcal{F}_t$

Indep lemma & MGF of normal to compute

Let $N_1 = \kappa\int_t^T (\hat{B}_s - \hat{B}_t)\, ds \sim N(0, \sigma_1^2)$

Let $N_2 = \sigma(\hat{B}_T - \hat{B}_t) \sim N(0, \sigma^2(T-t))$

Correlation : $\tilde{E}\, N_1 N_2 = r \kappa\, \hat{E}\left((\hat{B}_T - \hat{B}_t) \int_t^T (\hat{B}_s - \hat{B}_t)\, ds \right)$

$$= r \kappa \int_t^T \hat{E}\left((\hat{B}_T - \hat{B}_t)(\hat{B}_s - \hat{B}_t) \right) ds$$

& compute

Q6 Forward price of div paying stock

$$dS = \alpha S\,dt + \sigma S\,dW - a S_t\,dt$$ dividends $\downarrow$ rate

$$dX = \Delta_t\,dS_t + r(X - \Delta_t S_t)\,dt + \Delta_t\,a S_t\,dt$$

Under RNM:

$$d\hat{S} = r S_t\,dt + \sigma S_t\,d\hat{W}_t$$ (No dividends)

$$d\hat{S} = (r-a) S_t\,dt + \sigma S_t\,d\hat{W}$$ (With dividends)

FC: Pays $S_T - K$ at time T

Price at time t: $\dfrac{1}{D_t}\,\tilde{E}_t\left(D_T\,(S_T - K)\right)$

$$= e^{-r(T-t)}\,\tilde{E}_t\,(S_T - K)$$

$$= \cancel{e^{-r(T-t)}\,\tilde{E}_t S_T} - e^{-r(T-t)} K$$

Compute $\tilde{E}_t S_T = \tilde{E}_t S_t \exp\left(\left(r-a-\frac{\sigma^2}{2}\right)(T-t) + \sigma(\tilde{W}_T - \tilde{W}_t)\right)$

$$= S_t \exp\left(\left(r-a-\frac{\sigma^2}{2}\right)(T-t)\right) \tilde{E}_t \exp\left(\sigma(\tilde{W}_T - \tilde{W}_t)\right)$$

$$= S_t \exp\left(\left(r-a-\frac{\sigma^2}{2}\right)(T-t) + \frac{\sigma^2}{2}(T-t)\right)$$

$$\boxed{\Rightarrow V_t = \tilde{E}_t(\ldots) = \exp(-a(T-t))\, S_t - e^{-r(T-t)} K}$$

Price of FC of div paying stock.

FC of non div paying stock

$$S_t - e^{-r(T-t)} K$$

Replication: Buy $e^{-q(T-t)}$ shares at time t & borrow from bank