Q1 : Security on $\left(S_T^\gamma - K\right)^+$

RNM $\longrightarrow$ ( )

$$dS_t = r S_t \, dt + \sigma S \, d\widetilde{W} \qquad \left(\text{under } \widetilde{P}\right)$$

$\widetilde{W}$ → BM under $\widetilde{P}$

$$S_t^\gamma = S_0^\gamma \exp\left(\gamma\left(r - \frac{\sigma^2}{2}\right)t + \sigma\gamma \widetilde{W}_t\right)$$

$$S_T^\gamma = S_t^\gamma \exp\left(\gamma\left(r - \frac{\sigma^2}{2}\right)\underbrace{(T-t)}_{\tau} + \sqrt{\tau}\,\sigma\gamma\,\frac{\left(\widetilde{W}_T - \widetilde{W}_t\right)}{\sqrt{\tau}}\right)$$

$$V_t = e^{-r(T-t)}\,\tilde{E}_t\left(\underbrace{S_t^{\gamma}}_{\mathcal{F}_t \text{ meas.}} \exp\left(\gamma\left(r-\frac{\sigma^2}{2}\right)\tau + \gamma\sigma\sqrt{\tau}\,\underbrace{\left(\frac{\hat{W}_T-\hat{W}_t}{\sqrt{\tau}}\right)}_{N(0,1) \text{ ind of } \mathcal{F}_t}\right) - K\right)^+$$

$$\Rightarrow V_t = f(t, S_t).$$

$$f(t,x) = e^{-r\tau} \int_{y\in\mathbb{R}} \left[x^{\gamma} \exp\left(\gamma\left(r-\frac{\sigma^2}{2}\right)\tau + \gamma\sigma\sqrt{\tau}\,y\right) - K\right]^+ \frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy$$

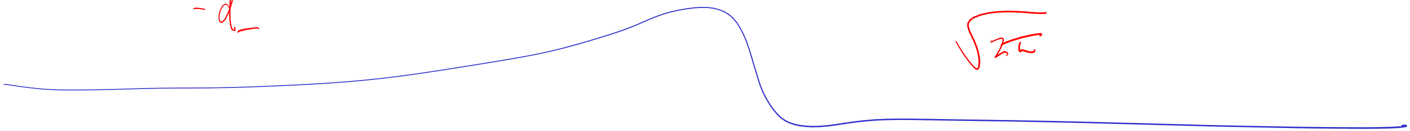
Solve $x^{\gamma} \exp\left(\gamma\left(r-\frac{\sigma^2}{2}\right)\tau + \gamma\sigma\sqrt{\tau}\right) = K \leftarrow -d$

split int as $\displaystyle\int_{-d_-}^{\infty} x^{\gamma}\, e^{\left(\gamma\left(r-\frac{\sigma^2}{2}\right)\tau+\gamma\sigma\sqrt{\tau}y - r\tau - y^2/2\right)} \frac{dy}{\sqrt{2\pi}}$

$-r(\quad)$ ✓ (fine)

Simplifies to $\displaystyle x^{\gamma}\int_{-d_-}^{\infty} \exp\left(-\frac{1}{2}\left(y^2-2\gamma\sigma\sqrt{\tau}y+2\tau\left(r(1-\gamma)+\frac{\sigma^2\gamma}{2}\right)\right)\right)\frac{dy}{\sqrt{2\pi}}$

$$= x^{\gamma} \int_{-d_-}^{\infty} e^{-\frac{1}{2}\left(y - \gamma\sigma\sqrt{\tau}\right)^2} \frac{dy}{\sqrt{2\pi}} + \text{const}$$



HW 12 Q2: Stock: $dS = \alpha_t S_t\, dt + \sigma_t S\, dW_t$

(σ – not random).

Find AFP of European Call.

1 Find a formula for S_t

$$\cancel{S_t = S_0 \exp\left(\left(r - \frac{\sigma_t^2}{2}\right)t + \sigma_t \widetilde{W}_t\right)} \leftarrow \text{(Only when } \sigma \text{ is time indep).}$$

(a) $$dS_t = r S_t\,dt + \sigma_t S_t\,d\widetilde{W}_t \qquad (\text{Valid under } \widetilde{P})$$

$$\rightarrow \frac{dS}{S} = r\,dt + \sigma_t\,d\widetilde{W}_t$$

Compute $$d(\ln S_t) \overset{\text{Ito}}{=} \frac{dS_t}{S_t} + \frac{1}{2}\left(\frac{-1}{S_t^2}\right) d[S_t, S_t]$$

$$= r\,dt + \sigma_t\,d\widetilde{W}_t - \frac{1}{2}\sigma_t^2\,dt$$

$$= \left(r - \frac{\sigma_t^2}{2}\right) dt + \sigma_t \, d\tilde{W}$$

$$\Rightarrow \ln\left(\frac{S_T}{S_0}\right) = \int_0^T \left(r - \frac{\sigma_t^2}{2}\right) dt + \int_0^T \sigma_t \, d\tilde{W}_t$$

$$\Rightarrow S_T = S_0 \exp\left(\int_0^T \left(r - \frac{\sigma_s^2}{2}\right) ds + \int_0^T \sigma_s \, d\tilde{W}_s \right)$$

$$\Rightarrow S_t = S_0 \exp\left(\int_0^t \left(r - \frac{\sigma^2}{2}\right) ds + \int_0^t \sigma \, d\tilde{W}_s \right)$$

$$\boxed{\Rightarrow S_T = S_t \exp\left(\int_t^T \left(r - \frac{\sigma^2}{2}\right) ds + \int_t^T \sigma \, d\tilde{W}_s \right)}$$

② $V_t = e^{-r\tau}\tilde{E}_t\left((S_T - K)^+\right)$

$\sum \sigma_{t_i}(W_{t_{i+1}} - W_{t_i})$

$$= e^{-r\tau}\tilde{E}_t\left(\left(S_t \exp\Big(\underbrace{\int_t^T \left(r - \frac{\sigma_s^2}{2}\right) ds + \int_t^T \sigma_s \, d\tilde{W}_s}_{Y}\Big) - K\right)^+\right)$$

Q: Is Y ind of $\mathcal{F}_t$? $\longrightarrow$ YES!

Want to apply ind lemma & compute this.

Need the dist of Y $\longrightarrow$ Claim Y is normally dist.

Find $\tilde{E} Y = \int_t^T \left(r - \frac{\sigma_s^2}{2}\right) ds$

$$Var(Y) = \tilde{E}\left(Y - EY\right)^2 = \tilde{E}\left(\int_t^T \sigma_s \, d\tilde{W}_s\right)^2$$

$$= \int_t^T \sigma_s^2 \, ds \quad \cancel{2}$$