Lecture ?? (4/18).

Please Enable Video if you can.

Last time: Know ① Compute RNM (1 stock 1 Bank)

② Know RNP formula & how to compute

(Main rule: Write all asset/security prices in terms of $\tilde{W}$ & use the fact that $\tilde{W}$ is a B.M. under $\tilde{P}$).

9.5. **The Martingale Representation Theorem.**

(i.e M is a mg & $\forall t$, $E M_t^2 < \infty$

Theorem 9.23. *If M_t is a square integrable martingale with respect to the Brownian filtration, then there exists a* predictable process *D such that $\boldsymbol{E}\int_0^t D_s^2\,ds < \infty$ and*

$$M_t = M_0 + \int_0^t D_s\,dW_s\,.$$

(M_0 is not random).

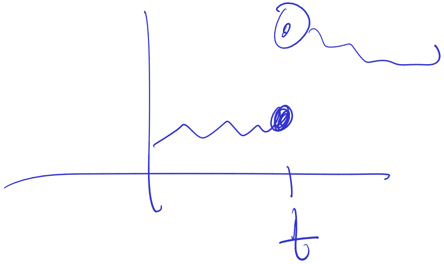
Remark 9.24. A *square integrable* martingale is a martingale for which $\boldsymbol{E}M_t^2 < \infty$ for all t.

Remark 9.25. For our purposes, think of a *predictable process* as a left continuous and adapted process.

370: A predictable process is a process + X_n is $\mathcal{F}_{n-1}$ meas.

420: A predictable process is one for which

$$\lim_{s \to t^-} X_s = X_t$$



Note: We know Ito int wrt BM are mg's. Mg rep $\Rightarrow$ the converse.

Theorem 9.26. *Consider the one stock market form Theorem 9.17.* $dS_t = \alpha_t S_t\,dt + \sigma_t S_t\,dW_t$

(1) *Any* $\tilde{\mathbf{P}}$ *martingale is the discounted wealth of a self financing portfolio (i.e. converse of Theorem 9.5 holds)*

(2) *Any security with an* $\mathcal{F}_T$*-measurable payoff is replicable, and so Theorem 9.7 holds for any* $\mathcal{F}_T$*-measurable function* V_T.

(3) *The risk neutral measure is unique.*

RNP

370: Dis wealth is mg $\iff$ self fin.

420: Thm 9.5: Self fin $\implies$ Disc wealth is a mg.

Mg rep thm $\implies$ converse.

Pf of (1) Assume X_t is process & $D_t X_t$ is a $\tilde{P}$ mg.

NTS X_t = wealth of a self fin port.

$\Leftrightarrow$ i.e. NTS $\exists$ an $\mathcal{F}_t$ adapted process Δ_t $\ni$

$$dX_t = \Delta_t \, dS_t + R_t (X_t - \Delta_t S_t) \, dt \leftarrow \text{Want.}$$

$\left(R_t \to \text{interest rate at time } t, \; D_t = \exp\left(-\int_0^t R_s \, ds\right) \right.$

$\left. \Leftrightarrow dD_t = -R_t D_t \, dt \;\&\; D_0 = 1 \right)$

① Know $D_t X_t$ is a $\tilde{P}$ mg.

Use mg rep thm (under $\tilde{P}$) to generate

$$\exists \; \Gamma_t \text{ (adapted)} \ni D_t X_t = D_0 X_0 + \int_0^t \Gamma_s \, d\tilde{W}_s.$$

$$\boxed{\Leftrightarrow\ d(D_t X_t) = \underbrace{\Gamma_t}\, d\widetilde{W}_t \quad \longleftarrow \text{Have} - - \ (*)}$$

(2) Want

$$dX_t = \Delta_t dS_t + R_t(X_t - \Delta_t S_t)\,dt$$

$$= \Delta_t(\underbrace{\alpha_t} S_t dt + \sigma_t S_t \underbrace{dW_t}_{\text{make } d\widetilde{W}}) + R(X_t - \Delta_t S_t)\,dt$$

$$= \Delta_t(\underbrace{R_t} S_t dt + \underline{\sigma_t S_t}\, d\widetilde{W}) + R_t(X_t - \Delta_t S_t)\,dt$$

$$\boxed{\Rightarrow dX_t = \Delta_t \sigma_t S_t d\widetilde{W} + R_t X_t\,dt} \longleftarrow \text{Want}.$$

Scratch: Assume what we want (i.e assume $\underbrace{dX_t = \Delta_t S_t d\tilde{W} + R_t X_t dt}$)

& compute $d(D_t X_t)$:

$$\boxed{d(D_t X_t) = D_t dX_t + X_t dD_t + \overbrace{d[X, D]_t}^{0}}$$

$$= D_t(\Delta_t \sigma_t S_t d\tilde{W} + \cancel{R_t} X_t dt) + X_t(-\cancel{R_t D_t} dt)$$

$$\boxed{\Rightarrow d(DX_t) = \underbrace{D_t \Delta_t \sigma_t S_t} d\tilde{W}. \quad **}$$

Actual proof: Know $\exists\, \Gamma_t$ (mg rep thm) $\ni$ $d(D_t X_t) = \Gamma_t \, d\tilde{W}$.

Choose $\Delta_t = \dfrac{\Gamma_t}{D_t \sigma_t S_t}$

Work backward through the above calculation (Scratch in green)

& get $dX_t = \Delta_t \, dS_t + R_t (X_t - \Delta_t S_t)\, dt$

$\Rightarrow X =$ wealth of a self fin port! QED.

Pf of ②: Say a sec pays V_T at time T

(V_T is $\mathcal{F}_T$ - meas)

NTS: Security is replicable.

Pf: Define X_t by $\underline{D_t X_t} = \underbrace{\tilde{E}_t(D_T V_T)}$

i.e. let $X_t = \frac{1}{D_t} \tilde{E}_t(D_T V_T)$.

<u>NOTE</u> $D_t X_t$ is a $\tilde{P}$ - mg!

$\left(\because\ \tilde{E}_s(D_t X_t) = \tilde{E}_s \hat{E}_t(D_T V_T) \overset{\text{tower}}{=} \tilde{E}_s(D_T V_T) = D_s X_s\right)$

By this part ① $\Rightarrow$ $D_t X_t$ is the disc wealth of a self fin port

$\Rightarrow$ X_t is the wealth of a self fin port.

Also $\underline{\underline{X_T}} = \frac{1}{D_T} \tilde{E}_T(D_T V_T) = \underline{\underline{V_T}}$

$\Rightarrow$ Sce is replicable QED.