HW 12 Q1

$$\text{Market} \begin{cases} \to \text{Stock GBM} \\ \to \text{Bank} \to \text{int rate } r \end{cases}$$

Security payoff $(S_T^\gamma - K)^+ \to$ Price this sec.

$$V_t = \frac{1}{e^{-rt}} \tilde{E}_t \left(e^{-rT} (S_T^\gamma - K)^+ \right) = e^{-r(T-t)} \underbrace{\tilde{E}_t (S_T^\gamma - K)^+}$$

Compute this.

$$dS_t = \alpha S_t \, dt + \sigma S_t \, dW_t$$

Useful trick to compute $\tilde{E}$: $\quad dS_t = \underline{r S_t \, dt} + \sigma S_t \, \underbrace{d\hat{W}_t}_{\text{BM under } \hat{P}}$

i.e. Under $\tilde{P}$, S is a GBM

with mean return rate r (= interest rate)

& vol σ (same as before)

$$\Rightarrow S_t = S_0 \exp\left(\left(\alpha - \frac{\sigma^2}{2}\right) t + \sigma W_t \right)$$ $\leftarrow$ useful under P

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma \tilde{W}_t \right) \leftarrow \text{useful under } \tilde{P}$$

$$\Rightarrow S_T = S_t \exp\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\left(\tilde{W}_T - \tilde{W}_t\right) \right)$$

$$\Rightarrow \left(S_T^\gamma - K\right)^+ = \left(S_t^\gamma \exp\left(\gamma\left(r - \frac{\sigma^2}{2}\right)(T-t) + \gamma\sigma\left(\tilde{W}_T - \tilde{W}_t\right) \right) - K \right)^+$$

$$\Rightarrow \tilde{E}_t\left(S_T^\gamma - K\right) = \tilde{E}_t\left(\underbrace{S_t^\gamma}_{\mathcal{F}_t\text{-meas}} \exp\left(\gamma\left(r - \frac{\sigma^2}{2}\right)(T-t) + \gamma\sigma\underbrace{\left(\tilde{W}_T - \tilde{W}_t\right)}_{\substack{\text{ind of } \mathcal{F}_t \\ \& \sim N(0,T-t)}} \right) - K \right)^+$$

Use indep lemma & compute!

Q4 $W \to$ 2D BM $W = (W^1, W^2)$

$$dX_t = \beta\, dt + \sin(\theta_t)\, dW^1_t + \cos(\theta_t)\, dW^2_t$$

Goal: Find ω's many equiv measures under which X is a BM!

Girsanov $\longrightarrow$ $d\tilde{W} = \sigma\, dt + 1\, dW$ $\tilde{W}$ is a BM under $\tilde{P}$

$$Z_T = \exp\left(-\int_0^T \sigma_t\, dW_t - \frac{1}{2}\int_0^T \sigma_t^2\, dt\right) \qquad d\tilde{P} = Z_T\, dP$$

Let $dB = \sin(\theta_t)\, dW_t^1 + \cos(\theta_t)\, dW_t^2$

$$\boxed{\Rightarrow dX_t = \beta_t\, dt + dB_t}$$

($\boxed{\text{If } B \text{ is a B.M.}}$ can Girsanov!)

Claim: B is a BM.

$\boxed{① \ B \text{ is a cts mg}}$ ✓

② $d[B,B]_t = \sin^2\theta_t \underbrace{d[W^1,W^1]_t}_{dt} + \cos^2\theta_t \underbrace{d[W^2,W^2]_t}_{dt} = dt$!!

$+ 2 \sin\theta_t \cos\theta_t \underbrace{d[W^1,W^2]_t}_{0}$

Levy $\Rightarrow$ B is a B.M.!

Going back: $dX_t = \beta_t\,dt + \underbrace{dB_t}_{\text{B.M.}}$

One measure $\tilde{P}$ under which X is a BM is given by

$$d\tilde{P} = Z_T\,dP \quad , \quad Z_T = \exp\left(-\int_0^T \beta_t\,dB_t - \frac{1}{2}\int_0^T \beta_t^2\,dt\right)$$

How do you find more measures $\tilde{P}$ under which X is a B.M.

Let $dB^1 = \sin\theta\,dW^1 + \cos\theta\,dW^2$ (same as B above)

Q: Can you find a process $\underline{\underline{B^2}}$ such that $(\underline{B^1}, \underline{B^2})$ is a $\underline{2D \quad B.M.}$

① Once you find B^2, Use the 2D Girsanov theorem to construct measures $\underline{\tilde{P}}$

$$Y = (X, m) \qquad dY^1 = d\underline{\underline{X}} = \beta \, dt + 1 \, d\underline{\underline{B^1}}$$

$$dY^2_t = \underline{\underline{(\text{anything})}} \, dt + 1 \, d\underline{\underline{B^2}}$$

Find $\tilde{P}$ to make Y a 2D BM $\Rightarrow X$ is a 1D B.M.

(2) Find B^2 :

Write $dB^2 = \gamma \, dW^1_t + \delta \, dW^2_t$

Need $d[B^2, B^2] = dt$

& $d[B^2, B^1] = 0 \, dt$

$\Big\}$ 2 equns for γ & δ.

Solve & conclude!