4/15 Lecture ?? (maybe 25?)

PLEASE ENABLE VIDEO IF YOU CAN

Need to be able to compute $\tilde{E}_t(D_T V_T)$

No nice formula

RNP Formula: $V_t = \frac{1}{D_t} \tilde{E}_t(D_T V_T)$

9.3. **Constructing Risk Neutral Measures.** Suppose the market has only one stock whose price process satisfies

Bank pays interest rate R_t

$$dS_t = \alpha_t S_t\,dt + \sigma_t S_t\,dW_t\,.$$

(α, σ both processes)

Theorem 9.17. *The (unique) risk neutral measure is given by $d\tilde{\boldsymbol{P}} = Z_T\,d\boldsymbol{P}$, where*

$$Z_T = \exp\Big(-\int_0^T \theta_t\,dW_t - \frac{1}{2}\int_0^T \theta_t^2\,dt\Big)\,, \qquad \boxed{\theta_t = \frac{\alpha_t - R_t}{\sigma_t}\,.}$$

$\leftarrow$ Market price of risk.

Proposition 9.18. *The stock price satisfies*

$$\to dS_t = R_t S_t\,dt + \sigma_t S_t\,d\tilde{W}\,,$$

where $\tilde{W}$ is a Brownian motion under the risk neutral measure.

Last time: $d\tilde{W} = \theta\,dt + dW$

Found $d\tilde{P}$ using Girsanov ($\Rightarrow$ Under $\tilde{P}$, $\underbrace{\tilde{W} \text{ is a BM!}}$)

$\to$ Pf: Know $dS_t = \alpha_t S_t\,dt + \sigma_t S_t\,dW_t$

$$= \alpha_t S_t\,dt + \sigma_t S_t\,(-\theta\,dt + d\tilde{W})$$

$$= \cancel{\alpha S\, dt} + \sigma S \left(\frac{R - \cancel{\alpha}}{\sigma}\, dt + d\tilde{W} \right)$$

$$= R_t S_t\, dt + \sigma_t S_t\, d\tilde{W}$$

Useful Under P: $\quad dS = \alpha_t S_t\, dt + \sigma_t S_t\, dW$

Useful under $\tilde{P}$: $\quad dS = R_t S_t\, dt + \sigma_t S_t\, d\tilde{W}$

9.4. **Black Scholes Formula revisited.**

- Suppose the interest rate $R_t = r$ (is constant in time).
- Suppose the price of the stock is a GBM(α, σ) (both α, σ are constant in time).

$dS = \alpha S\,dt + \sigma S\,dW$

Theorem 9.19. *Consider a security that pays $V_T = g(S_T)$ at maturity time T. The arbitrage free price of this security at any time $t \leqslant T$ is given by $f(t, S_t)$, where*

$$f(t,x) = \int_{-\infty}^{\infty} e^{-r\tau} g\Big(x \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)\tau + \sigma\sqrt{\tau}\, y\Big)\Big) \frac{e^{-y^2/2}dy}{\sqrt{2\pi}}, \qquad \tau = T - t. \tag{7.4}$$

Remark 9.20. This proves Proposition 7.8.

Pf: Use RNM. Know under $\tilde{P}$, $dS_t = r S\,dt + \sigma S\,d\tilde{W}_t$

($\tilde{W}$ is a BM under $\tilde{P}$) $\Rightarrow S = \text{GBM}(r, \sigma)$ under $\tilde{P}$.

$$\Rightarrow S_t = S_0 \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)t + \sigma \tilde{W}_t\Big)$$

$$S_T = S_0 \exp\Big(\Big(r - \frac{\sigma^2}{2}\Big)T + \sigma \tilde{W}_T\Big)$$

$$\Rightarrow \frac{S_T}{S_t} = \exp\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\left(\tilde{W}_T - \tilde{W}_t\right)\right)$$

$$\Rightarrow \underline{S_T} = \underline{S_t}\, \underbrace{\exp\left(\left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\left(\tilde{W}_T - \tilde{W}_t\right)\right)}$$

② RNP Formula:

$$V_t = \text{AFP at time } t = \frac{1}{D_t} \tilde{E}_t (D_T V_T) \qquad (D_t = e^{-rt})$$

$$= e^{-r(T-t)} \tilde{E}_t V_T = e^{-r(T-t)} \tilde{E}_t\, g(\underline{S_T})$$

$$= e^{-r\tau}\,\tilde{E}_t\, g\left(\underbrace{S_t}_{\mathcal{F}_t\text{-meas}} \exp\left[\left(r-\frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\,\underbrace{\frac{\tilde{W}_T-\tilde{W}_t}{\sqrt{\tau}}}_{\substack{\text{ind of } \mathcal{F}_t \\ \sim N(0,1)}}\right]\right)$$

desired formula!

$$\boxed{V_t \overset{\text{indep lemma}}{=} e^{-r\tau}\int_{y=-\infty}^{\infty} g\left(S_t \exp\left[\left(r-\frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\,y\right]\right)\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy}$$

Theorem 9.21 (Black Scholes Formula)**.** *The arbitrage free price of a European call with strike K and maturity T is given by:*

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)) \tag{7.5}$$

where

$$d_\pm(\tau,x) \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{\tau}}\Big(\ln\Big(\frac{x}{K}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)\tau\Big)\,, \tag{7.6}$$

and

$$N(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy\,, \tag{7.7}$$

is the CDF of a standard normal variable.

Remark 9.22. This proves Corollary 7.9.

Pf: AFP of call at time t = $V_t = c(t, S_t) = e^{-r\tau}\tilde{E}_t(S_T - K)^+$

By 9.19: $c(t,x) = e^{-r\tau}\displaystyle\int_{-\infty}^{\infty} \Big(\underbrace{x\exp\Big[\Big(r - \frac{\sigma^2}{2}\Big)\tau + \sigma\sqrt{\tau}\,y\Big] - K}_{\text{when is this} > 0\,?}\Big)^+ e^{-y^2/2}\frac{dy}{\sqrt{2\pi}}$

Solve $\quad x\, e^{\left(r-\frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\, y} = K$

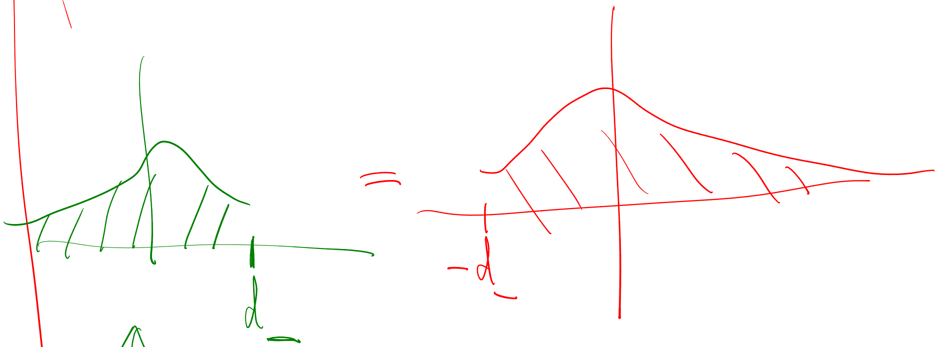
$$\iff \left(r-\frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\, y = \ln\left(\frac{K}{x}\right)$$

$$\iff y = \frac{-1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{x}{K}\right) + \left(r-\frac{\sigma^2}{2}\right)\tau\right) = -d_-$$

$$\Rightarrow c(t,x) = e^{-r\tau}\int_{-d_-}^{\infty}\left(x\, e^{\left(r-\sigma^2/2\right)\tau + \sigma\sqrt{\tau}\, y} - K\right)^{\cancel{+}} e^{-y^2/2}\,\frac{dy}{\sqrt{2\pi}}$$

$$= \underbrace{\int_{-d_-}^{\infty} x\, e^{(r-\frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}\, y - r\tau} \cdot e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}} - K e^{-r\tau} \underbrace{\int_{-d_-}^{\infty} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}}_{N(d_-)}$$

Simplify



$P(N(0,1) \leq d_-) = N(d_-)$

a

$$= -K e^{-r\tau} N(d_-) + \int_{-d_-}^{\infty} x\, e^{-\frac{1}{2}\left(\sigma^2\tau - 2\sigma\sqrt{\tau}\, y + y^2\right)} \frac{dy}{\sqrt{2\pi}}$$

$$= -K e^{-r\tau} N(d_-) + x \int_{-d_-}^{\infty} e^{-\frac{1}{2}\left(y - \sigma\sqrt{\tau}\right)^2} \frac{dy}{\sqrt{2\pi}}$$

Put $z = y - \sigma\sqrt{\tau}$

$dz = dy$

$$= \quad {''} \quad + x \int_{-\underbrace{(d_- + \sigma\sqrt{\tau})}_{d_+}}^{\infty} e^{-z^2/2} \frac{dz}{\sqrt{2\pi}} = x N(d_+) - K e^{-r\tau} N(d_-)$$

9.5. **The Martingale Representation Theorem.**

Theorem 9.23. *If M_t is a square integrable martingale with respect to the Brownian filtration, then there exists a* predictable process *D such that $\boldsymbol{E}\int_0^t D_s^2\,ds < \infty$ and*

$$M_t = M_0 + \int_0^t D_s\,dW_s\,.$$

Remark 9.24. A *square integrable* martingale is a martingale for which $\boldsymbol{E}M_t^2 < \infty$ for all t.

Remark 9.25. For our purposes, think of a *predictable process* as a left continuous and adapted process.

Theorem 9.26. *Consider the one stock market form Theorem 9.17.*

(1) Any $\tilde{\boldsymbol{P}}$ martingale is the discounted wealth of a self financing portfolio (i.e. converse of Theorem 9.5 holds)
(2) Any security with an $\mathcal{F}_T$-measurable payoff is replicable, and so Theorem 9.7 holds for any $\mathcal{F}_T$-measurable function V_T.
(3) The risk neutral measure is unique.

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Existence of RNM $\iff$ No arb

Exist & Uniq $\iff$ No arb & completeness