Lecture 8 (2/4). Please ENABLE VIDEO if you can

Last time: Cond exp. → NO explicit formula

→ SAME properties as in the disc case

(examples and w/o being a lot harder)

Definition 5.5. $\boxed{\boldsymbol{E}_t X}$ is the unique *random variable* such that:

(1) $\boldsymbol{E}_t X$ is $\mathcal{F}_t$-measurable.

(2) For every $A \in \mathcal{F}_t$, $\displaystyle\int_A \boldsymbol{E}_t X \, d\boldsymbol{P} = \int_A X \, d\boldsymbol{P}$

(i.e. $E(1_A E_t X) = E(1_A X)$)

Remark 5.6. Choosing $A = \Omega$ implies $\boldsymbol{E}(\boldsymbol{E}_t X) = \boldsymbol{E} X$.

Proposition 5.7 (Useful properties of conditional expectation)**.**

(1) If $\alpha, \beta \in \mathbb{R}$ are constants, X, Y, random variables $\boldsymbol{E}_t(\alpha X + \beta Y) = \alpha \boldsymbol{E}_t X + \beta \boldsymbol{E}_t Y$.

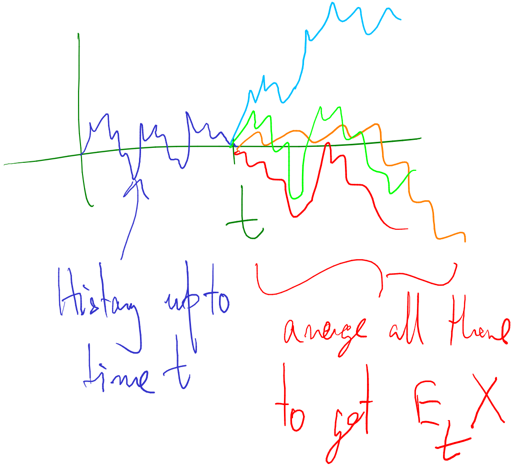
(2) If $X \geqslant 0$, then $\boldsymbol{E}_t X \geqslant 0$. Equality holds if and only if $X = 0$ almost surely.

(3) (Tower property) *If $0 \leqslant s \leqslant t$, then $\boldsymbol{E}_s(\boldsymbol{E}_t X) = \boldsymbol{E}_s X$.*

(4) If X is $\mathcal{F}_t$ measurable, and Y is any random variable, then $\boldsymbol{E}_t(XY) = X \boldsymbol{E}_t Y$.

(5) If X is $\mathcal{F}_t$ measurable, then $\boldsymbol{E}_t X = X$ (follows by choosing $Y = 1$ above).

(6) If Y is independent *of $\mathcal{F}_t$, then $\boldsymbol{E}_t Y = \boldsymbol{E} Y$.*



Remark 5.8. These properties are exactly the same as in discrete time.

Lemma 5.9 (Independence Lemma)**.** *If X is $\mathcal{F}_t$ measurable, Y is independent of $\mathcal{F}_t$, and $f = f(x, y) \colon \mathbb{R}^2 \to \mathbb{R}$ is any function, then*

$$\boldsymbol{E}_t f(X, Y) = g(X)\,, \quad \text{where} \quad g(y) = \boldsymbol{E} f(X, y).$$

Remark 5.10. If p_Y is the PDF of Y, then $\boldsymbol{E}_t f(X, Y) = \int_{\mathbb{R}} f(X, y)\, p_Y(y)\, dy$.

Then $E_t f(X, Y) =$ "average Y & leave X alone"

$= g(X)$ where $g(x) = E f(x, \overset{\text{RV}}{Y})$

Example 5.11. If X, Y are two independent standard normal random variables, find $\boldsymbol{E} e^{iXY}$.

Option 1: X & Y ind normal $\Rightarrow$ Joint PDF of (X, Y) is

$$\frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

$$\Rightarrow E e^{iXY} = \int_{\mathbb{R}^2} e^{ixy} \cdot e^{-(x^2+y^2)/2} \frac{dx\,dy}{2\pi}.$$

& compute this integral

Option 2: Nice trick using the indep Lemma.

X, Y indep.

Let $\mathcal{F} = \sigma(X) =$ all events that can be observed using the RV X

i.e. $\{X > 0\} \in \sigma(X)$. $\{X \in [1, 2]\} \in \sigma(X)$

Obs: X is meas wrt $\sigma(X)$
& Y is ind of $\sigma(X)$ $\Big\}$ $\to$ Use indep lemma!!

Notation $E_t(Z) = E(Z \mid \mathcal{F}_t)$

Compute: $E(e^{iXY}) \overset{\text{tower}}{=} E\left(E\left(e^{iXY} \mid \sigma(X)\right)\right)$

Notation

Notation

$= E\left(E\left(e^{iXY} \mid X\right)\right)$

By the indep lemma,

$$E\left(e^{iXY} \mid X\right) = \underline{\underline{\text{avg } Y}},\ \overbrace{\text{leave } X \text{ alone}}$$

$$= g(X) \quad \text{where} \quad g(x) = E\left(e^{ixY}\right)$$

$$= \text{char fn of std nrml}$$

$$= \underline{\underline{e^{-x^2/2}}}$$

$$= e^{-X^2/2}$$

$$\Rightarrow E\left(e^{iXY} \mid X\right) = e^{-X^2/2}$$

$$\Rightarrow E e^{iXY} = E\left(E\left(e^{iXY} \mid X\right)\right) = E e^{-X^2/2}$$

$$= \int_{-\infty}^{\infty} e^{-x^2/2} \, e^{-x^2/2} \, \frac{dx}{\sqrt{2\pi}}$$

$$= \underbrace{\int_{-\infty}^{\infty} e^{-x^2/2 \cdot \left(\frac{1}{\sqrt{2}}\right)^2} \frac{dx}{\sqrt{2\pi \left(\frac{1}{\sqrt{2}}\right)^2}}}_{1} \left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}.$$

5.4. **Martingales.**

M_t is $\mathcal{F}_t$ meas $\forall t$

Definition 5.12. An adapted process M is a martingale if for every $0 \leqslant s \leqslant t$, we have $\boldsymbol{E}_s M_t = M_s$.

Remark 5.13. As with discrete time, a martingale is a fair game: stopping based on information available today will not change your expected return.

Proposition 5.14. $\boxed{\textit{Brownian motion is a martingale.}}$

Proof.

(Analog in disc time : ξ_n iid, $E\xi_n = 0$, Set $X_{n+1} = X_n + \xi_{n+1}$

i.e $X_n = \sum_1^n \xi_k$

$X \to$ discr time RW.

Know X is a mg (from 370).

Check BM is a mg:

Know $W_0 = 0$, $W_t - W_s \sim N(0, t-s)$

& $W_t - W_s$ is ind of $\mathcal{F}_s$.

NTS - BM is a mg

i.e. NTS $\forall\ 0 \leq s \leq t$, $E_s W_t = W_s$

Pf: $E_s W_t = E_s(W_t - W_s + W_s)$

$= E_s(W_t - W_s) + E_s W_s$

$= E(W_t - W_s) + W_s$

$= 0 + W_s$ $= \boxed{W_s}$

$\Rightarrow$ B.M. is a mg.

Q: Is W_t^2 a mg? NO

Is W_t^3 a mg? NO

On HW: Compute $E_s W_t^3$