HW2 Q2

$$E X_n = \mu_n$$

$$E(X_n-\mu_n)^2 = \sigma_n^2.$$

Want CLT.

$(\sigma_n \longrightarrow \sigma)$

Suppose 1: Say $\underline{\mu_n = 0}$ $\forall n$.

Copy CLT statement: $S_n = \frac{1}{\sqrt{n}}\sum_1^n X_k$

Want $\frac{S_n}{\sqrt{n}} \longrightarrow$ Normal. $N(0,\sigma^2)$ ~~$N(0,1)$~~

COPY the proof!! Find φ_{S_n}

Find φ_{X_n} first

Guess $\varphi_{X_n}(\lambda) = 1 + 0\lambda - \sigma_n^2 \lambda^2 + \cdots$

$$\varphi_{S_n}(\lambda) = E e^{i\lambda S_n} = \prod_{k=1}^{n} \left(1 - \frac{\sigma_k^2 \lambda^2}{2} + \cdots\right)$$

$$\varphi_{\frac{S_n}{\sqrt{n}}}(\lambda) = \varphi_{S_n}\left(\frac{\lambda}{\sqrt{n}}\right) = \prod_{k=1}^{n} \left(1 - \sigma_k^2 \frac{\lambda^2}{2n} + \underbrace{O\left(\lambda^3 / n^{3/2}\right)}_{\text{ignore.}}\right)$$

Q: $\sigma_k \xrightarrow{k\to\infty} \sigma$

$\Rightarrow$ find $\lim_{n\to\infty} \prod_{k=1}^{n} \left(1 - \frac{\sigma_k^2 \lambda^2}{2n} + O\left(\frac{\lambda^3}{n^{3/2}}\right)\right) = L$

$$\Rightarrow \ln L = \lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(1 - \frac{\sigma_k^2 \lambda^2}{2n} + O\left(\frac{\lambda^3}{n^{3/2}}\right)\right)$$

$$\boxed{\ln(1+x) \approx x}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(-\frac{\sigma_k^2 \lambda^2}{2n} + O\left(\frac{\lambda^3}{n^{3/2}}\right)\right)$$

$$Q: \lim_{n\to\infty} \frac{-\lambda^2}{2n} \sum_{k=1}^{n} \sigma_k^2 = -\frac{\lambda^2}{2} \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sigma_k^2$$

$$= -\frac{\lambda^2 \sigma^2}{2}$$

$$Q: \lim_{n\to\infty} \sum_{k=1}^{n} O\left(\frac{\lambda^3}{n^{3/2}}\right) \leq n\, O\left(\frac{\lambda^3}{n^{3/2}}\right) \leq \frac{C}{n^{1/2}} \longrightarrow 0$$

Gives $\frac{S_n}{\sqrt{n}} \longrightarrow N(0, \sigma^2)$

(in the case that $\mu_n = 0 \ \forall n$)

Q3:

Compute $E W_s^2$, $E W_t^2$, $E W_s W_t$

s t

Known

$$\left\{ \begin{array}{ll} ⓐ & W_t - W_s \sim N(0, t-s) \\ ⓑ & W_t - W_s \text{ ind of } \mathcal{F}_s \end{array} \right\}$$

→ Put $s = 0$

$W_0 = 0$

$\Rightarrow W_t - \cancel{W_0} \sim N(0, t - \cancel{0})$

$\Rightarrow \underline{W_t \sim N(0, t)}$