Lecture 5 (Jan 28)

Please ENABLE VIDEO If you can.

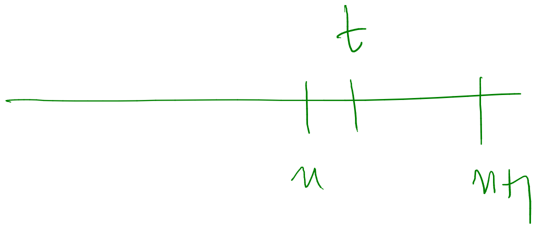
Last time → Constant B.M.

Brownian Motion → cts time R.W.

R.W.: $X_n = \sum_{1}^{k} \xi_k$, $\xi_k \to$ iid, $E\xi_k = 0$, $E\xi_k^2 = 1$

$\hookrightarrow X_{n+1} = X_n + \xi_{n+1}$.

$$Y_t = Y_n + (t-n)\,\xi_{n+1} \qquad t \in [n, n+1)$$



Flip coins every ε seconds

Let $Y^{\varepsilon}_t = \sqrt{\varepsilon}\, Y_{t/\varepsilon}$ (RW with step size $\sqrt{\varepsilon}$ & coin flips occuring every ε seconds)

B. M. $\rightarrow$ cts time RW $\rightarrow$ send $\varepsilon \rightarrow 0$

Define $$W_t = \lim_{\varepsilon \to 0} Y^{\varepsilon}_t = \lim_{\varepsilon \to 0} \sqrt{\varepsilon}\, Y_{t/\varepsilon}.$$

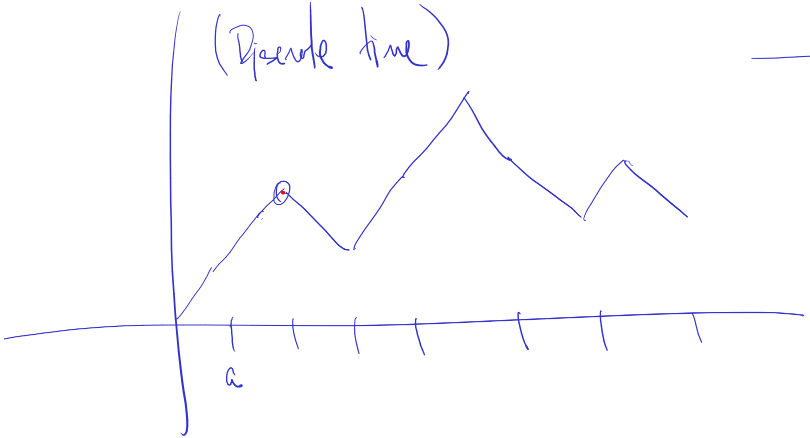
- If t, s are multiples of ε: $Y_t^\varepsilon - Y_s^\varepsilon \sim \sqrt{\varepsilon} \sum_{i=1}^{(t-s)/\varepsilon} \xi_i \xrightarrow{\varepsilon \to 0} \mathcal{N}(0, t-s)$. (CLT, last time)
- $Y_t^\varepsilon - Y_s^\varepsilon$ only uses coin tosses that are "after s", and so independent of Y_s^ε.

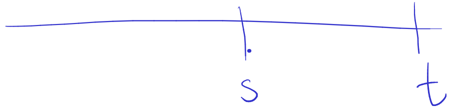
Definition 5.2. Brownian motion is a $\boxed{\textit{continuous process}}$ such that: $W_0 = 0$

(1) $W_t - W_s \sim \mathcal{N}(0, t-s)$,
(2) $W_t - W_s$ is independent of $\mathcal{F}_s$.

Remark 5.3. We will define $\mathcal{F}_s$ shortly. Intuitively, $\mathcal{F}_s$ is the set of all events that are "known" at time s.

Pickup





For any $s < t$ expect ~~the~~ BM to change direction infinitely often between times s & t.

Also ~~know~~ Expect B.M to be a cts fn of time.

Claim: BM is always a ~~te~~ cts fn of time

& With Prob 1, BM is not diff anywhere (wrt time)

Finance : Standard model for stock price

(Geometric B.M.) $$S_t = S_0 \exp\left(\left(\alpha - \frac{\sigma^2}{2}\right) t + \sigma W_t \right)$$

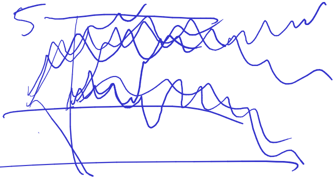
$\alpha \to$ mean return rate

$\sigma \to$ volatility

$\boxed{W \to \text{Brownian motion!!}}$

5.2. **Sample space, measure, and filtration.**

- Discrete time: Sample space $\Omega = \{(\omega_1, \dots, \omega_N)\}$. ω_i = outcome of i^{th} coin toss $\}$.
- View $(\omega_1, \dots, \omega_N)$ as the trajectory of a random walk.
- Continuous time: Sample space $\Omega = C([0, \infty))$ (space of continuous functions).
  - ▷ It's infinite. No probability mass function!
  - ▷ Mathematically impossible to define $\boldsymbol{P}(A)$ for *all* $A \subseteq \Omega$.



PMF

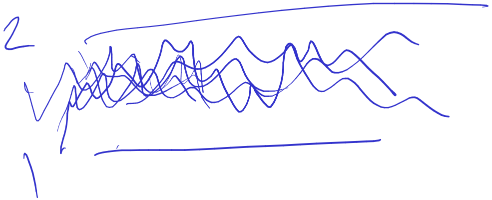
$p : \Omega \to \mathbb{R}$, $\boxed{\sum_{\omega \in \Omega} p(\omega) = 1}$

Def $P(A) = \sum_{\omega \in A} p(\omega)$

Discrete time

$\Omega = \{(\omega_1, \dots, \omega_N) \mid \omega_i \to i^{th}$ coin toss$\}$ → can't easily generalize.

→ $\mathcal{F}_n$ = all events that can be observed using only coin tosses before n.



$$X_{n+1} = X_n + \underbrace{\omega_{n+1}}_{n+1^{th}\text{ coin toss.}} \qquad (RW)$$

$(\omega_1, \dots \omega_N) \longrightarrow$ tells me the trajectory of the RW

Cts time: Sample space $= \{$ paths BM can take $\}$

$= \{$ all cts fns, domain $[0,\infty)$ & target $\mathbb{R} \}$

$= C([0,\infty))$ (intuition

- Restrict our attention to $\mathcal{G}$, a subset of some sets $A \subseteq \Omega$, on which $\boldsymbol{P}$ can be defined.
  - ▷ $\mathcal{G}$ is a σ-algebra. (Closed countable under unions, complements, intersections.)
- $\boldsymbol{P}$ is called a *probability measure* on $(\Omega, \mathcal{G})$ if:
  - ▷ $\boldsymbol{P}\colon \mathcal{G} \to [0,1]$, $\boldsymbol{P}(\emptyset) = 0$, $\boldsymbol{P}(\Omega) = 1$.
  - ▷ $\boldsymbol{P}(A \cup B) = \boldsymbol{P}(A) + \boldsymbol{P}(B)$ if $A, B \in \mathcal{G}$ are disjoint.
  - ▷ If $A_n \in \mathcal{G}$, $\boldsymbol{P}\left(\bigcup_1^\infty A_n\right) = \lim_{n\to\infty} \boldsymbol{P}(A_n)$.
- Random variables are *measurable* functions of the sample space:
  - ▷ Require $\{X \in A\} \in \mathcal{G}$ for every "nice" $A \subseteq \mathbb{R}$.
  - ▷ E.g. $\{X = 1\} \in \mathcal{G}$, $\{X > 5\} \in \mathcal{G}$, $\{X \in [3,4)\} \in \mathcal{G}$, etc.
  - ▷ Recall $\{X \in A\} = \{\omega \in \Omega \mid X(\omega) \in A\}$.

elts of $\mathcal{G}$.

If $A_1, A_2, \dots$ are ∞'ly many disj ~~sets~~ events

then $P\left(\bigcup_{n=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k)$

$\Omega \Rightarrow$ Sample space

$\mathcal{G}$ is a σ- algebra on Ω if

① $\mathcal{G}$ is a non-empty collection of subsets of Ω

② $\phi \in \mathcal{G}$, $\Omega \in \mathcal{G}$.

③ If $A, B \in \mathcal{G}$, then A^c, $A \cup B$, $A \cap B \in \mathcal{G}$.

④ If $A_1, A_2, \ldots \in \mathcal{G}$ then $\bigcup_{1}^{\infty} A_k \in \mathcal{G}$.

Say $X : \Omega \longrightarrow \mathbb{R}$.

Q: $P(X > 0) = P(\overbrace{\{\omega \in \Omega \mid X(\omega) > 0\}})$

$= P(\{X > 0\})$