Lecture 4 (Jan 26)

Please ENABLE VIDEO if you can.

Last time: CLT $\rightarrow$ $X_n \rightarrow$ iid, $EX_n = 0$, $EX_n^2 = 1$

then $\frac{S_n}{\sqrt{n}} \longrightarrow \underbrace{N(0,1)}$

$\left(S_n = \sum_{k=0}^{n} X_k\right)$

n

Review from Prob

① Multivariate Normal → density, covariance matrix. etc

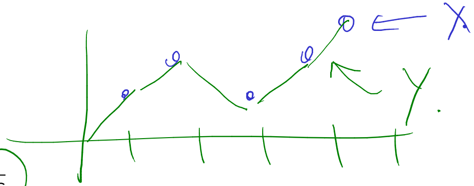
② Linear transformation of Normal → Normal.
(& can compute mean & cov)

③ limit of Normal → Normal (will say more in class).

5. Stochastic Processes.

5.1. Brownian motion.

- Discrete time: Simple Random Walk.
  - ▷ $X_n = \sum_1^n \xi_i$, where ξ_i's are i.i.d. $\boldsymbol{E}\xi_i = 0$, and $\operatorname{Range}(\xi_i) = \{\pm 1\}$.
- Continuous time: Brownian motion.
  - ▷ $Y_t = X_n + (t-n)\xi_{n+1}$ if $t \in [n, n+1)$.
  - ▷ Repeat more frequently: Flip a coin every ε seconds, and take a step of size $\sqrt{\varepsilon}$.
  - ▷ Rescale: $Y_t^\varepsilon = \sqrt{\varepsilon} Y_{t/\varepsilon}$. (Chose $\sqrt{\varepsilon}$ factor to ensure $\operatorname{Var}(Y_t^\varepsilon) \approx t$.)
  - ▷ Let $W_t = \lim_{\varepsilon \to 0} Y_t^\varepsilon$.



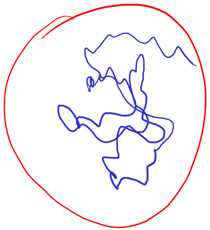
Wiener Process

cts time RW.

Definition 5.1 (Brownian motion)**.** The process W above is called a Brownian motion.

▷ Named after Robert Brown (a botanist).
▷ Definition is intuitive, but not as convenient to work with.

$W_t =$ "$\lim_{\varepsilon \to 0}$" $\left(\sqrt{\varepsilon}\, Y_{t/\varepsilon}\right)$



$Y_{\frac{t}{\varepsilon}}$

$Y_t \longrightarrow \lfloor t \rfloor$ steps of a RW (ctsly joined).

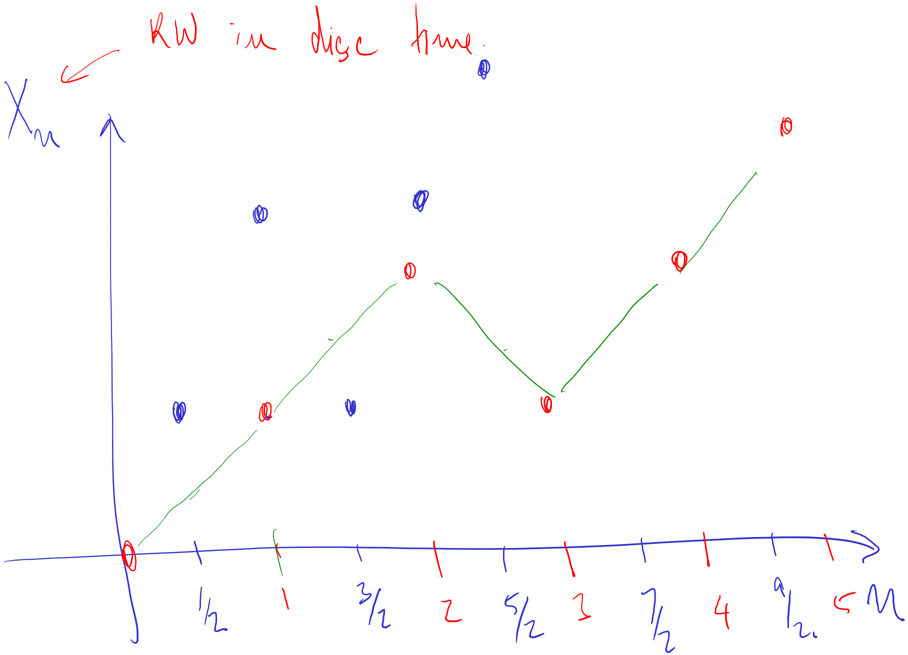
$Y_{t/\varepsilon} \longrightarrow \lfloor t/\varepsilon \rfloor$ steps of a RW (ctsly joined)

step size 1.

$\sqrt{\varepsilon}\, Y_{t/\varepsilon} \longrightarrow \lfloor \frac{t}{\varepsilon} \rfloor$ steps of a RW (ctsly joined).
step size $\sqrt{\varepsilon}$

$\xi_k \to$ iid.

$$X_n = \sum_{k=1}^{n} \xi_k$$



Want RW in cts time

Want to keep variance at time 1 constant

(1) Flip coins every second : Var at time $1 = 1$

(step size 1)

(2) Flip coins every $\frac{1}{2}$ secs : Var at time $1 = 1 + 1 = 2$.

step size : $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{\sqrt{2}}$ Var at time $1 = \frac{1}{n}$

(4) 4 " " $\frac{1}{n}$ secs : Var at time $1 = 2$.

step size $\frac{1}{2^{n/2}}$ 2^n $\longrightarrow$ " " " $1 = 1$.

To keep variance constant : decrease size of the step as we decrease the interval between coin flips.

$$\xi_1 = \begin{cases} a & \text{prob } \frac{1}{2} \\ -a & \text{prob } \frac{1}{2}. \end{cases} \qquad (a = \text{step size})$$

$\mathrm{Var}(\xi_1) = a^2$. Want Var after 2 steps $= 1 \Leftrightarrow 2a^2 = 1 \Leftrightarrow a = \frac{1}{\sqrt{2}}$

Let $\underset{\color{red}{\sim\sim}}{W_t} = \lim_{\varepsilon \to 0} \sqrt{\varepsilon}\, Y_{t/\varepsilon} = \lim_{\varepsilon \to 0} Y^{\varepsilon}_t .$

Q: Does this lim exist? $\Rightarrow$ Yes (Hard)

Q: Can we say something about the lim? $\Rightarrow$ Yes (CLT!)

- If t, s are multiples of ε: $\boxed{Y_t^\varepsilon - Y_s^\varepsilon \sim \sqrt{\varepsilon} \sum_{i=1}^{(t-s)/\varepsilon} \xi_i \xrightarrow{\varepsilon \to 0} \mathcal{N}(0, t-s).}$
- $Y_t^\varepsilon - Y_s^\varepsilon$ only uses coin tosses that are "after s", and so independent of Y_s^ε.

Definition 5.2. Brownian motion is a *continuous process* such that:

(1) $W_t - W_s \sim \mathcal{N}(0, t-s)$,
(2) $W_t - W_s$ is independent of $\mathcal{F}_s$.

Remark 5.3. We will define $\mathcal{F}_s$ shortly. Intuitively, $\mathcal{F}_s$ is the set of all events that are "known" at time s.

$$W_t = \lim_{\varepsilon \to 0} \sqrt{\varepsilon}\, Y_{t/\varepsilon}.$$

s, t mult of ε. Prefered

$$W_t \approx \sqrt{\varepsilon}\, Y_{t/\varepsilon}$$

$$W_s \approx \sqrt{\varepsilon}\, Y_{s/\varepsilon}$$

$$W_t - W_s \approx \sqrt{\varepsilon}\left(Y_{t/\varepsilon} - Y_{s/\varepsilon}\right) = \sqrt{\varepsilon}\left(X_{t/\varepsilon} - X_{s/\varepsilon}\right)$$

$$= \sqrt{\varepsilon}\left(\sum_{k=1}^{t/\varepsilon}\xi_k - \sum_{k=1}^{s/\varepsilon}\xi_k\right)$$

$$= \sqrt{\varepsilon}\underbrace{\sum_{k=s/\varepsilon}^{t/\varepsilon}\xi_k}_{\text{sum of } \frac{t-s}{\varepsilon} \text{ iid RV's}} \xrightarrow{\varepsilon \to 0}$$

$$= \sqrt{t-s}\,\frac{1}{\left(\frac{t-s}{\varepsilon}\right)^{1/2}}\sum_{k=s/\varepsilon}^{t/\varepsilon}\xi_k.$$

$$\text{CLT} \xrightarrow[\varepsilon \to 0]{} N(0,1).$$

$$\longrightarrow \sqrt{t-s}\; N(0,1) = N(0, t-s).$$

Bottom line: Expect $W_t - W_s \sim N(0, t-s)$.