Lecture 3 (1/24). Please ENABLE VIDEO if you can

Last time: CF (characteristic fn)

$$\varphi_X(\lambda) = E\, e^{i\lambda X} \qquad (\varphi_X : \mathbb{R} \to \mathbb{C})$$

MGF: $M_X(\lambda) = E\, e^{\lambda X}$

$$EX = -i\,\varphi_X'(0) \qquad \left(\text{b/c} \quad \partial_\lambda \varphi_X(\lambda) = E\left(iX\underbrace{e^{i\lambda X}}\right) \Rightarrow \varphi_X'(0) = i\,EX \right)$$

$$EX^2 = -\varphi_X''(0)$$

Let X, Y be two random variables.

Theorem 4.13. *The following are equivalent.*

(1) *X and Y have the same distribution (PDF)*
(2) *X and Y have the same CDF.*
(3) *X and Y have the same characteristic function.*
(4) *X and Y have the same moment generating function.*

CDF of X: $F_X(x) = P(X \le x)$

i.e. $\forall \lambda \in \mathbb{R}$, $\varphi_{X_n}(\lambda) \to \varphi_X(\lambda)$

Theorem 4.14. *A sequence of random variables $(X_n) \to X$ (in distribution) if and only if $\varphi_{X_n} \to \varphi_X$ pointwise.*

Theorem 4.15. *A sequence of random variables $(X_n) \to X$ (in distribution) if and only if $M_{X_n} \to M_X$ pointwise.*

key

Remark 4.16. The proofs of Theorem 4.13–4.15 are beyond the scope of this course; we will use them without proof.

Need another assumption

Note: If X & Y have the same pdf f

then $\varphi_X(\lambda) = E\, e^{i\lambda X} = \int_{\mathbb{R}} e^{i\lambda x} f(x)\,dx$

$\varphi_Y(\lambda) = E\, e^{i\lambda Y} = \int_{\mathbb{R}} e^{i\lambda y} f(y)\,dy$

equal!

$\Rightarrow$ X & Y have the same C.F.

((iii) $\Rightarrow$ X & Y have the same MGF)
(ii)

. Similarly

Rmk: Hard part of Thm 4.13 is showing same CF/MGF

$\Rightarrow$ same dist/CDF.!

Note: We say $\varphi_{X_n} \longrightarrow \varphi_X$ pointwise if for every $\lambda \in \mathbb{R}$

$$\varphi_{X_n}(\lambda) \longrightarrow \varphi_X(\lambda).$$

Proof of Theorem 4.7. ~~4~~ CLT.

Thm 4.7: X_n iid. $\overbrace{EX_n = 0}$, $\overbrace{EX_n^2 = 1}$

$$S_n = \sum_{1}^{n} X_k.$$

std normal.

CLT: $\dfrac{S_n}{\sqrt{n}} \xrightarrow{\text{dist}} N(0,1)$

Strategy: Show $CF\left(\dfrac{S_n}{\sqrt{n}}\right) \xrightarrow{\text{ptwise}} CF(N(0,1))$

<u>Step 1</u>: Compute the form of the CF of X_k.

Know ① $E X_k = 0 = -i\,\varphi'_{X_k}(0) \Rightarrow \underline{\underline{\varphi'_{X_k}(0) = 0}}$

Know ② $E X_k^2 = 1 = -\varphi''_{X_k}(0) \Rightarrow \varphi''_{X_k}(0) = \underline{\underline{-1}}$

Know ③ $\varphi_{X_k}(0) = E e^{i 0 X_k} = \underline{\underline{1}}$

Expect $\varphi_{X_k}(\lambda) = 1 + \underbrace{0}_{\varphi'_{X_k}(0)} \lambda \underbrace{- \frac{1}{2}}_{\varphi''_{X_k}(0) = -1} \lambda^2 + O(\lambda^3)$

$$\boxed{\Rightarrow \varphi_{X_k}(\lambda) = 1 - \frac{\lambda^2}{2} + O(\lambda^3)}$$

Step 2: Find φ_{S_n}:

$$\varphi_{S_n}(\lambda) = E\, e^{i\lambda S_n} = E\, e^{i\lambda \sum_1^n X_k}$$

$$= E\left(\prod_{k=1}^{n} e^{i\lambda X_k}\right) \overset{\text{indep}}{=\!=} \prod_{k=1}^{n} E e^{i\lambda X_k}$$

$$= \prod_{k=1}^{n} \varphi_{X_k}(\lambda) = \left(1 - \frac{\lambda^2}{2} + O(\lambda^3)\right)^n$$

Step 3: $\varphi_{\left(\frac{S_n}{\sqrt{n}}\right)}(\lambda) = E\, e^{i\lambda S_n/\sqrt{n}}$

$$= E\, e^{i\left(\frac{\lambda}{\sqrt{n}}\right) S_n} = \varphi_{S_n}\left(\frac{\lambda}{\sqrt{n}}\right)$$

$$= \left[1 - \frac{\lambda^2}{2n} + O\left(\frac{\lambda^3}{n^{3/2}}\right)\right]^n$$

Step 4: Compute $\lim_{n\to\infty} \varphi_{\frac{S_n}{\sqrt{n}}}(\lambda)$

Know $\lim_{n\to\infty} \varphi_{\frac{S_n}{\sqrt{n}}}(\lambda) = \lim_{n\to\infty} \left(1 - \frac{\lambda^2}{2n} + O\left(\frac{\lambda^3}{n^{3/2}}\right)\right)^n$

$$= \lim_{n\to\infty} \exp\left(n \ln\left(1 - \underbrace{\frac{\lambda^2}{2n} + O\left(\frac{\lambda^3}{n^{3/2}}\right)}_{x} \right)\right)$$

$$e^x = \exp(x)$$

$$= \lim_{n\to\infty} \exp\left(\cancel{n} \left(-\frac{\lambda^2}{2\cancel{n}} + O\left(\frac{\lambda^3}{n^{3/2}}\right)\right)\right)$$

Know $\underbrace{\ln(1+x) \approx 0}\, + \frac{1}{1} x + O(x^2)$

$$\ln(1+x) \approx x$$

$$= \exp\left(\frac{-\lambda^2}{2}\right)$$

$$= \text{char fn of } N(0,1)$$

(evaluated at λ). QED.

5. **Stochastic Processes.**

5.1. **Brownian motion.**

- Discrete time: Simple Random Walk.
  - ▷ $X_n = \sum_1^n \xi_i$, where ξ_i's are i.i.d. $\boldsymbol{E}\xi_i = 0$, and $\operatorname{Range}(\xi_i) = \{\pm 1\}$.
- Continuous time: Brownian motion.
  - ▷ $Y_t = X_n + (t-n)\xi_{n+1}$ if $t \in [n, n+1)$.
  - ▷ Repeat more frequently: Flip a coin every ε seconds, and take a step of size $\sqrt{\varepsilon}$.
  - ▷ Rescale: $Y_t^\varepsilon = \sqrt{\varepsilon} Y_{t/\varepsilon}$. (Chose $\sqrt{\varepsilon}$ factor to ensure $\operatorname{Var}(Y_t^\varepsilon) \approx t$.)
  - ▷ Let $W_t = \lim_{\varepsilon \to 0} Y_t^\varepsilon$.

Definition 5.1 (Brownian motion)**.** The process W above is called a Brownian motion.

▷ Named after Robert Brown (a botanist).

▷ Definition is intuitive, but not as convenient to work with.

- If t, s are multiples of ε: $Y_t^\varepsilon - Y_s^\varepsilon \sim \sqrt{\varepsilon} \sum_{i=1}^{(t-s)/\varepsilon} \xi_i \xrightarrow{\varepsilon \to 0} \mathcal{N}(0, t-s)$.
- $Y_t^\varepsilon - Y_s^\varepsilon$ only uses coin tosses that are "after s", and so independent of Y_s^ε.

Definition 5.2. Brownian motion is a *continuous process* such that:

(1) $W_t - W_s \sim \mathcal{N}(0, t-s)$,
(2) $W_t - W_s$ is independent of $\mathcal{F}_s$.

Remark 5.3. We will define $\mathcal{F}_s$ shortly. Intuitively, $\mathcal{F}_s$ is the set of all events that are "known" at time s.